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INTRODUCTION TO LATTICE GAUGE THEORY

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CONTENTS

1) INTRODUCTION

This is a lightening quick introduction to Lattice Gauge Theory. Many topics are omitted and those covered may at times be starved of details. I shall make an effort to remedy this problem by pointing you to appropriate references (A brief list of recommended reading for background material is [1] [2] [3] [4] [5] [6]). The aim here is to give you an overview of the field.

The lattice formulation of Quantum Field Theory (QFT) can be exploited in many ways. 1) We can derive the lattice Feynman rules and carry out weak coupling perturbation expansions. The lattice then serves as a manifestly gauge invariant regularization scheme, albeit one that is more complicated than standard continuum schemes. 2) Strong coupling expansions: these give us useful qualitative information, but unfortunately no hard numbers [7] . 3) The lattice theory is amenable to numerical simulations by which one calculates the long distance properties of a strongly interacting theory from first principles. The observables are measured as a function of the bare coupling g and a gauge invariant cut-off $\approx \frac{1}{a}$, where a is the lattice spacing. The continuum (physical) behavior is recovered in the limit $a \to 0$, at which point the lattice artifacts go to zero. This is the more powerful use of lattice formulation, so in these lectures I will focus on setting up the theory for the purpose of numerical simulations to get hard numbers.

The numerical techniques used in Lattice Gauge Theories have their roots in statistical mechanics, so it is important to develop an intuition for the interconnection between quantum mechanics and statistical mechanics. This will be the emphasis of the first lecture. In the second lecture, I will review the essential ingredients of formulating QCD on the lattice and discuss scaling and the continuum limit in lecture 3. In the last lecture I will summarize the status of some of the main results. I will also mention the bottlenecks and possible directions for research.

1.1) Connection between QM and Statistical Mechanics

The first step is to establish the connection between the partition function and the transition amplitude in quantum mechanics. For this consider the simple two state system described by the Hamiltonian

$$H \qquad a_0 \left(\sigma_3 \qquad \lambda\sigma_1\right) \tag{1.1}$$

in Euclidean time τ where λ is small and the eigenstates $|\uparrow\rangle$, $|\downarrow\rangle$ of σ_3 are used as basis states. The time evolution of a state - the quantum amplitude $\psi(f,i)$ - is given by

$$\psi(f,i) \qquad \langle f|e^{-H\tau}|i\rangle \quad . \tag{1.2}$$

To set up the path integral, divide the time interval τ into N intervals, $\tau = N\delta\tau$. Then

$$\begin{aligned}\psi(f,i) &= \langle f|e^{-H(\delta\tau + \delta\tau + \ldots + \delta\tau)}|i\rangle \\ &\approx \sum_{s_n}\cdots\sum_{s_1}\langle f|e^{-H\delta\tau}|s_{N-1}\rangle\langle s_{N-1}|e^{-H\delta\tau}|s_{N-2}\rangle\ldots\ldots\langle s_1|e^{-H\delta\tau}|i\rangle \quad (1.3)\end{aligned}$$

where $\sum_s |s\rangle\langle s| = 1$ is a complete set of states. Eqn. (1.3) is the path integral representation of the quantum amplitude. The sum over states corresponds to all possible paths (configurations) of the system. The correct weight factor e^{-S} where S is the action arises from the accumulation of $e^{-H\delta\tau}$ factors along the path [8] .

The quantum mechanical amplitude for finite time of propagation can be written as a product of Feynman kernels K

$$\psi(f,i) = \sum_{s_n}\cdots\sum_{s_1} K(f,s_{n-1})\,K(s_{n-1},s_{n-2})\ldots\ldots K(s_2,s_1)\,K(s_1,i) \qquad (1.4)$$

with a sum over intermediate states. $K(s_f,s_i)$ is the transition amplitude to go from state s_i at time τ to the state s_f at time $\tau+\delta\tau$. In the limit of small $\delta\tau$, we can solve for the kernel $K(s_i,s_j)$

$$\begin{aligned}K(s_i,\,s_j) \equiv \langle s_i|e^{-H\delta\tau}|s_j\rangle &\approx \langle s_i|1-\delta\tau a_0(\sigma_3 - \lambda\sigma_1)|s_j\rangle \\ &= \begin{pmatrix} 1-a_0\delta\tau & a_0\lambda\delta\tau \\ a_0\lambda\delta\tau & 1+a_0\delta\tau\end{pmatrix} \qquad (1.5)\end{aligned}$$

In the Schrödinger representation both K and ψ are classical c-numbers. Using eqn (1.5), ψ can also be written as

$$\psi(f,i) \approx \langle f|e^{-H\delta\tau}|s_n\rangle \begin{pmatrix} 1-a_0\delta\tau & a_0\lambda\delta\tau \\ a_0\lambda\delta\tau & 1+a_0\delta\tau\end{pmatrix}^{N-2} \langle s_1|e^{-H\delta\tau}|i\rangle \qquad (1.6)$$

The first and last factors are row and column vectors and depend on the choice of the final and initial states f,i respectively.

To make the connection with statistical mechanics, regard the kernel $K(s_i,s_j)$ as an interaction between neighboring spins and rewrite the amplitude in eqn. (1.4) as

$$\sum_{s_n=\pm 1}\cdots\sum_{s_i=\pm 1} e^{-T(f,s_n)}\,e^{-T(s_n,s_{n-1})}\cdots e^{-T(s_1,i)} \qquad (1.7)$$

with spins f and i held fixed. To reproduce the path integral to $O(\delta\tau)$, we need an expression for T which reproduces the matrix elements in eqn.(1.5). By inspection the answer is

$$T(s_i,s_j) = \beta\frac{(s_i-s_j)^2}{4} + h\frac{(s_i+s_j)}{2}. \qquad (1.8)$$

where in a SM system with s_i the spin at site i, β would be the nearest neighbor coupling and h the magnetic interaction. In eqn.(1.7), setting $f = i$ and summing over i gives

$$Z = \sum_i \sum_{s_n} \cdots \sum_{s_1} e^{-T(i,s_n)} \cdots e^{-T(s_1,i)} \qquad (1.9a)$$

the partition function for a $1-$*dimensional* Ising system with periodic boundary conditions with T given in eqn.(1.8).

One purpose of defining the path integral in terms of T, eq.(1.7) was to introduce the transfer matrix $e^{-\mathcal{T}}$. $\mathcal{T}$ is the Hamiltonian for discrete Euclidean time measured in lattice units; $\mathcal{H} = \lim_{\delta\tau \to 0} \frac{\mathcal{T}}{\delta\tau}$. The definition of β and h has to reproduce the matrix elements of eqn.(1.5) and satisfy the definition $\langle f|e^{-\mathcal{T}}|i\rangle \equiv K(f,i)$. A necessary condition for this is that $\mathcal{T}$ be proportional to $\delta\tau$. This is satisfied by

$$Z = \sum_i \sum_{s_n} \cdots \sum_{s_1} \exp\Big(- \delta\tau \sum_i [-a_0\lambda \frac{(s_i - s_j)^2}{4} + a_0 \frac{(s_i + s_j)}{2}]\Big) \ . \qquad (1.9b)$$

Note that each term of the sum in the exponent is the discretized Euclidean version of the Hamiltonian in eqn.(1.1). This association of the transfer matrix as the discrete time evolution operator is an alternate route to go directly from QM to SM.

This completes a demonstration of the first connection; the relation of the partition function to the QM amplitude $\sum_i \psi(i,i)$ and the path integral. In fact the partition function in SM is equivalent to the generating functional in QFT.

Let me now show the following relationships: 1) the free energy in SM corresponds to the ground state energy in QM; 2) correlation functions $\Longleftrightarrow$ time ordered expectation values and 3) correlation length $\Longleftrightarrow$ reciprocal of the mass-gap. These are true in general even though they are derived for the simple 2-state system.

For small λ, eqn. (1.9) for $f = i$ reduces to

$$Z = e^{a_0 \tau} + e^{a_0 \tau}$$

in the limit $\delta\tau \to 0$, $N\delta\tau \to \tau$. Z is the partition function for the two state system. For $\tau \to \infty$,

$$\frac{1}{\tau} \ln Z = \textit{free energy} = a_0 \qquad (1.10)$$

since only the ground state contributes to the sum in the partition function.

Next consider the time ordered product

$$\langle i|T[\sigma_3(\tau)\sigma_3(1)]|i\rangle$$

where $\sigma_3(i)$ is an operator in the Heisenberg representation. Using eqn. (1.3) we get

$$\begin{aligned} &\approx \sum_{s_n} \cdots \sum_{s_1} \langle f|e^{-H\delta\tau}|s_n\rangle \ldots \sigma_3|s_r\rangle\langle s_r| \ldots \sigma_3|s_1\rangle\langle s_1|e^{-H\delta\tau}|i\rangle \\ &= \sum_{s_n} \cdots \sum_{s_1} s_r s_1 \, \langle f|e^{-H\delta\tau}|s_n\rangle\langle s_n| \ldots\ldots\ldots \langle s_1|e^{-H\delta\tau}|i\rangle \end{aligned}$$

since $|s_j\rangle$ are eigenstates of σ_3. Now it is straightforward to repeat the steps between eqns. (1.3) and (1.8) to get the second relation, correlation functions $\Longleftrightarrow$ Euclidean time ordered expectation values

$$\begin{aligned} \frac{\langle i|\mathrm{T}[\sigma_3(\tau)\sigma_3(1)]|i\rangle}{\langle i|e^{-H\tau}|i\rangle} &= \frac{1}{Z} \sum_{i} \sum_{s_n} \cdots \sum_{s_1} s_r s_1 \; e^{-T(i,s_n)} \ldots \; e^{-T(s_1,i)} \\ &\equiv \langle s_r s_1 \rangle \quad . \end{aligned} \tag{1.11}$$

To show the last relation, insert a complete set of states of $\mathcal{H}$ in eqn (1.3) rather than of σ_3. Let $|0\rangle, |1\rangle$ be the ground state and the excited state with enegy E_0, E_1 respectively. Then

$$\begin{aligned} \langle s_r s_1 \rangle &= \frac{\langle i|\mathrm{T}[\sigma_3(\tau)\sigma_3(1)]|i\rangle}{\langle i|e^{-H\tau}|i\rangle} \\ &= \frac{1}{e^{-E_0\tau} + e^{-E_1\tau}} \Big[\, |\langle 0|\sigma_3|0\rangle|^2 \, e^{-E_0\tau} + |\langle 1|\sigma_3|1\rangle|^2 \, e^{-E_1\tau} + \\ &\qquad |\langle 0|\sigma_3|1\rangle|^2 \left(e^{-E_0(\tau-\tau)-E_1\tau} + e^{-E_1(\tau-\tau)-E_0\tau} \right) \Big] \end{aligned}$$

In the limit of large τ, the leading terms on the right hand side are

$$\langle s_r s_1 \rangle \; \to \; |\langle 0|\sigma_3|0\rangle|^2 + |\langle 0|\sigma_3|1\rangle|^2 \, e^{-(E_1-E_0)\tau} \quad . \tag{1.12}$$

For a theory with a non zero mass gap $m \equiv (E_1 - E_0)$, the second term controls the exponential decay of the correlation functions at large distance. In SM, such an exponential decay is characterized by a correlation length ξ:

$$\frac{1}{\xi} \equiv m = (E_1 - E_0) \quad . \tag{1.13}$$

If the first term in eqn (1.12) is non-zero, then for $\tau \to \infty$ the correlations do not go to zero. The system is said to have long range order. In our simple example, the σ_3 term in $\mathcal{H}$ tends to align the spins *i.e.* acts like a magnetic field as shown in eqn (1.8). Its presence breaks the up down symmetry of the states (which exists for $\mathcal{H} \propto \sigma_1$). Thus $\langle 0|\sigma_3|0\rangle \neq 0$. So, in this simple example, σ_3 is an operator that explicitly breaks the symmetry.

In QM, interactions are characterized by non-zero off-diagonal matrix elements. If O is any operator that causes transitions between eigenstates of H, then

$$\langle i|O|j\rangle^2 \neq \langle i|O^2|i\rangle \quad . \tag{1.14a}$$

i.e. the system has quantum fluctuations. In SM, an analogous statement is applicable to correlation functions in the canonical ensemble

$$\langle O\rangle^2 \neq \langle O^2\rangle \tag{1.14b}$$

i.e. states with different value of O contribute to the sum in the partition function. Thus fluctuations are essential features of both quantum and statistical systems.

Having built up the formal analogy between QM and SM, let me next formulate and solve the harmonic oscillator problem as a warm-up exercise.

1.2) The Harmonic Oscillator

The essential concepts and necessary background that readers should be familiar with are given in chapters 3, 8, and 10 of Feynman and Hibbs [8].

The Lagrangian for a SHO in Euclidean time τ is

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \omega^2 x^2) \tag{1.15}$$

The quantum amplitude is given by the path integral

$$K(y,\tau,x,0) = \int_{paths} \mathcal{D}x(\tau)\, e^{-\int_0^\tau \mathcal{L}(x,\dot{x},t)dt} \tag{1.16}$$

i.e. each path from $(x,0)$ to (y,τ) contributes with weight e^{-S} where S is the action. For the SHO, the kernel can be solved for in a closed form

$$K(y,\tau,x,0) = \left(\frac{m\omega}{2\pi\sinh\omega\tau}\right)^{\frac{1}{2}} \exp\left\{-\frac{m\omega}{2\sinh\omega\tau}[(x^2+y^2)\cosh\omega\tau - 2xy]\right\} \quad . \tag{1.17}$$

To set up the problem for numerical analysis, consider the evolution for infinitesimal time ϵ

$$K(y,\epsilon,x,0) \approx \sqrt{\frac{m}{2\pi\epsilon}}\, \exp\left\{-\frac{m\epsilon}{2}\left[\left(\frac{y-x}{\epsilon}\right)^2 + \omega^2\frac{(x^2+y^2+xy)}{3}\right]\right\} \quad . \tag{1.18}$$

In analogy with eqn (1.7), regard K as a Boltzmann factor specifying the interaction between points at two adjacent times. This interpretation of the kernel as a probability can be used to generate the paths.

For the SHO we are able to write a closed form expression for K. In general this is not possible. One then uses the alternate way to go from QM to SM; by defining a partition

function with the Transfer matrix playing the role of the standard Boltzmann factor. For example note that by taking the discrete version of $\mathcal{L}$ given in eqn. (1.15) we reproduce the exponent in eqn. (1.18). In this discretization, Euclidean time is treated as an extra dimension. The kinetic energy term in $\mathcal{H}$ connects variables in this extra "time" dimension and the potential energy terms prescribe the spatial couplings. In general, a d-dimensional QM system is represented by a $(d+1)$-dimensional SM system. In the remaining part of these lectures, this will be the implicit route.

The only restriction to setting up the path integral for numerical simulations is that $\mathcal{K}$ should be interpretable as a probability. This is guaranteed if the action S is real and positive definite (more specifically bounded from below so that the measure is normalizable). If we can regard $\mathcal{K}$ as a probability then we can generate a series of co-ordinates x_n at time $(0+n\epsilon)$ for $n = 1, N$ with some starting x_0. This discrete series of points form a path on a lattice of spacing ϵ. The steps to simulate the path integral are

(a) choose some x_0 at $\tau = 0$;
(b) generate x_1 at $\tau = \epsilon$ with probability $\mathcal{K}$ given by eqn.(1.18);
(c) generate x_{i+1} from x_i as in step (b) for N time intervals where $(N\epsilon = \tau)$;
(d) repeat steps a) to c) infinite number of times.

The path integral is calculated from the infinite set of all possible paths between $(x, 0)$ and (y, τ) with their respective weight e^{-S} (shown graphically in Fig.1.1) . Note that changing the oscillator parameters m and ω does not change the paths, it changes the weight associated with a given path. This will remain a key concept as we go from a single particle system to N particles to QFT.

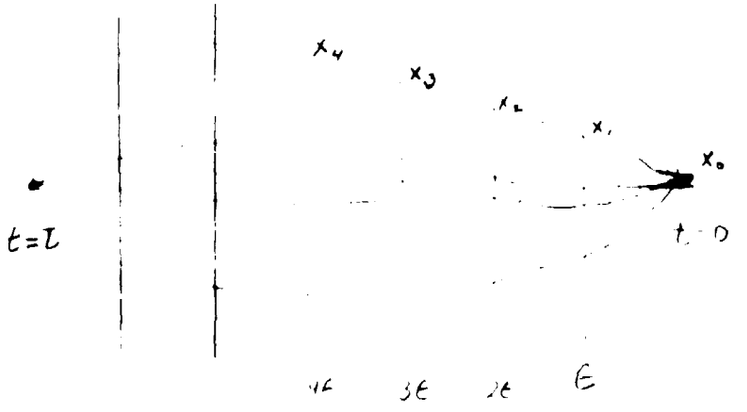

Fig.1.1: A set of illustrative paths for the SHO.

Given the paths, all quantum mechanical expectation values at some time $n\epsilon$ are

calculated as simple averages over the distribution of x_i:

$$\langle x_\tau^n \rangle \;=\; \frac{1}{N}\sum_{i=1}^{N} x_i(\tau) \qquad (1.19a)$$

or the 2−point correlation functions

$$\langle x_\tau x_{\tau+m\epsilon} \rangle \;=\; \frac{1}{N}\sum_{i=1}^{N} x_i(\tau)\; x_i(\tau + m\epsilon) \qquad (1.19b)$$

where the average is over the $N \to \infty$ paths (path integral) generated by the above procedure. The first average, eqn (1.19a), gives the expectation value in the ground state while the 2-point function, eqn (1.19b), (n-point functions in general) provide dynamical information.

If we choose the starting x_0 with a distribution $|\psi_0|^2$ *i.e.* the ground state probability distribution, then the expectation value in eqn (1.19a) is the ground state expectation value at all time slices (no τ dependence). Otherwise, if we start with a random distribution of x_0 which has a non-zero overlap with the ground state then we need to discard some number of time steps before measuring expectation values. In this latter case we rely on the exponential damping to remove the higher states. To calculate the expectation values in some excited state, we need to choose a distribution of x_0 that is orthogonal to all the lower states. Thus, the starting distribution of x_0 plays an important role in the evolution in two respects. 1) To project onto a definite eigenstate ψ, the distribution of starting x_0 has to be $|\psi(x)|^2$. 2) If the starting state is not an eigenstate then after a certain number of time steps (which depends on m, ω and ϵ) only the lowest state consistent with the symmetries of starting x_0 survives due to the exponential damping.

The above discussion implies that the distribution of x_i generated on any time slice is $|\psi_0|^2$ if the starting distribution of x_0 is ψ_0. This at first glance sounds fishy - we have so far been talking about the quantum amplitude and now suddenly there appears a probability. To understand why recall two facts - the law of composition of kernels (amplitudes)

$$K(y,x) \;=\; \int K(y,z)\; K(z,x)\; dz$$

and that the kernel is an evolution operator, so in the language of a complete set of energy eigenstates ϕ_i

$$\begin{aligned} K(y,x) \;&=\; \sum_i e^{-E_i\tau}\; \phi_i(y)\; \phi_i^*(x) \\ &=\; \sum_i e^{-E_i\tau}\; \phi_i(y) \int \phi_j^*(z)\phi_j(z)dz\; \phi_i^*(x) \end{aligned}$$

which shows that if we evolve an eigenstate ϕ, then the distribution of z at any intermediate time has weight $\phi^*\phi$. Thus the expectation values we calculate in eqn. (1.19) are the analogues of $\int x^n \phi^*(x)\phi(x)dx$.

In case you are wondering how the amplitude ψ will acquire the expected e^{-Er} behavior since none of the paths terminate (the distribution of x is the same on all time slices), let me outline how this damping is built up in an actual numerical algorithm. To select y from x using eqn (1.18) one procedure is to generate y with probability

$$P(y) \;=\; \exp\left[-\frac{1}{2}m\epsilon\left(\frac{x-y}{\epsilon}\right)^2\right] \quad , \tag{1.20}$$

and then accept or reject this y with probability

$$1 - \exp\left[-\frac{m\epsilon\omega^2}{2}\frac{(x^2+y^2+xy)}{3}\right] \quad . \tag{1.21}$$

To get the correct normalization, *i.e.* in order to generate y according to eqn (1.18), the accepted y have to be weighted with

$$\frac{1}{\sqrt{1+\frac{\omega^2\epsilon^2}{3}}}\;\exp\left[-\frac{m\omega^2\epsilon}{2}x^2\left(\frac{1+\omega^2\epsilon^2/12}{1+\omega^2\epsilon^2/3}\right)\right] \tag{1.22}$$

It is this last factor that builds up the e^{-Er}. As an exercise show that you need the weight factor in eqn (1.22) to get the correct normalization of eqn (1.18), once y is generated by eqns. (1.20) and (1.21).

2) FIELD THEORY ON THE LATTICE

2.1) $\lambda\phi^4$ Theory in 4-dimensions

In the previous two examples we saw how to set up the path integral and calculate the correlation functions for $0-d$ QM systems. Next, I am going to jump to the simplest field theory in 3+1 Euclidean dimensions -- scalar $\lambda\phi^4$. The action on the lattice is

$$S \;=\; \sum_x \left[\,-\,\kappa\,\phi^\dagger(x)\sum_{\pm\mu}\phi(x+\hat{\mu}) \;+\; |\phi(x)|^2 \;+\; \lambda(|\phi(x)|^2-1)^2\right] \tag{2.1}$$

which as stated before is the Boltzmann factor with which to generate configurations. Here we are going from QM to SM via the Transfer matrix. The standard continuum action is obtained with the following correspondence

$$\phi_c \;=\; \sqrt{2\kappa}\;\phi \tag{2.2a}$$

$$m^2 \;=\; \frac{(1-2\lambda)}{\kappa} - 8 \tag{2.2b}$$

$$\lambda_c \;=\; \frac{\lambda}{\kappa^2} \quad . \tag{2.2c}$$

ϕ is a real valued variable (in general an $n-tuplet$ for an $O(N)$ model) at each site on the lattice. A path (configuration) is defined by specifying the value of $\phi(x_1, x_2, x_3, x_4)$ at all points on the lattice. The path integral is calculated from the set of all possible configurations. Correlation functions (probes of physics) are calculated as simple averages over these configurations with their respective weights as discussed in section 3.1. The weight of any configuration in the functional integral is e^{-S}.

So conceptually and operationally it is very easy to set up the path integral. The bottleneck in simulations is that implementation can be (and is for QCD) prohibitively expensive. Some of the reasons are: 1) the need for large lattices, 2) small signal to noise ratio in observables and 3) slow evaluation of S especially when dynamical fermions are included. I hope these facts shall become clear in the following lectures.

A technical aside; the lattice need not be symmetric *i.e.* the lattice spacing in the space and time direction can be different. To get a field theory we are interested in the limit $\delta x \to 0$ and $\delta \tau \to 0$ (see section 3.1 for why). Thus one can consider two limiting processes: 1) The Hamiltonian limit; the spacing in the time direction is taken to zero first keeping the spatial spacing fixed. This gives a discrete Hamiltonian system with continuous time. The limit $\delta x \to 0$ is taken next. 2) the Euclidean limit; the spacing in the spatial and time directions is kept equal *i.e.* $\delta x = \delta \tau$, and the continuum limit is taken preserving this symmetry. Which is the better method to approach the continuum limit depends on the model and the particular observable being calculated. The physics is the same. To illustrate this consider a correlation length in a $2-d$ model. In the Euclidean approach ξ is approximately isotropic barring lattice artifacts for small ξ. As $\delta \tau$ is made small compared to δx, the constant ξ circle distorts into an increasingly eccentric ellipse. Measured in lattice units ξ seems very different in the two directions. However, converted to physical units it is again roughly isotropic (there are some quantum corrections).

In these lectures I will restrict the discussion to the symmetric Euclidean formulation unless otherwise specified. This Hamiltonian approach is explained in more detail in section 5 of Ref. [4].

2.2) QCD on the Lattice

The lattice is a $(3+1)$ dimensional Euclidean grid of side L with spacing a. Our goal is to define the field variables on the lattice and with them construct an action which reduces to the familiar continuum form $\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \gamma_\mu (\partial_\mu + igA_\mu) \psi$ in the limit $a \to 0$.

The fermion fields are functions of space time points and are defined at each site on the lattice. However, they are Grassmann variables for which we have no direct representation that is amenable to numerical simulation. The solution is to intergrate them out since the fermion action is bilinear. Unfortunately, the resulting coupling between the gauge degrees

of freedom becomes totally non-local. This, as will be discussed later, is a bottleneck in present calculations.

The gauge fields carry 4-vector Lorentz indices and mediate interactions so they are not associated with the sites. To define them, recall that in the continuum a fermion moving from site x to y in presence of gauge fields $A_\mu(x)$ ($\equiv \lambda_a \cdot A^a_\mu(x)$) picks up a path ordered phase

$$\psi(y) \;=\; \mathcal{P}\, e^{\int_x^y igA_\mu(x)dx_\mu}\; \psi(x) \quad . \tag{2.3}$$

This suggests that gauge fields live on links that connect sites on the lattice. So, with each link is associated a discrete version of the path ordered product

$$U(x, x+\hat{\mu}) \;\equiv\; U_\mu(x) \;=\; e^{iagA_\mu(x+\frac{\hat{\mu}}{2})} \tag{2.4}$$

where a is the lattice spacing. For concreteness, we shall define the field A_μ to exist at the midpoint of the link. Also, the path ordering in eqn. (2.3) specifies that

$$U(x, x-\hat{\mu}) \;=\; e^{-iagA_\mu(x-\frac{\hat{\mu}}{2})} \;=\; U^\dagger(x-\mu, x) \quad . \tag{2.5}$$

The dimension of U is specified by the representation of color SU(3) to which the fermions are assigned. Unless otherwise specified we shall assume that it is the fundamental representation. Then U is a 3×3 unitary matrix with determinant one.

The local gauge freedom acts on the sites. Under a local gauge transformation $V(x)$

$$\psi(x) \;\rightarrow\; V(x)\psi(x) \tag{2.6a}$$

$$\overline{\psi}(x) \;\rightarrow\; \overline{\psi}(x)V^\dagger(x) \tag{2.6b}$$

$$U_\mu(x) \;\rightarrow\; V(x)U_\mu(x)V^\dagger(x+\hat{\mu}) \tag{2.6c}$$

where $V(x)$ is in the same representation as the U *i.e.* it is a $SU(3)$ matrix.

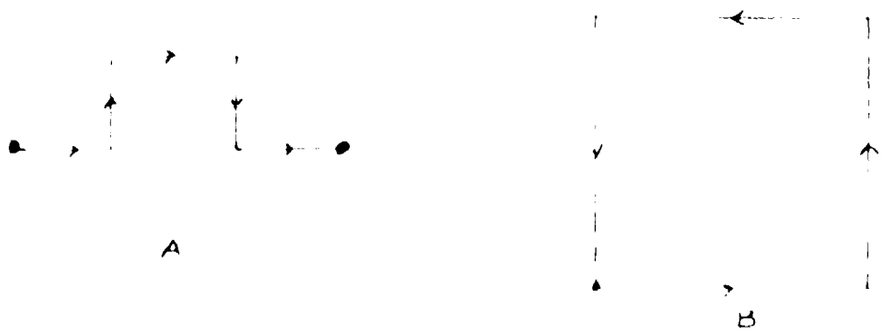

Fig.2.1: The two gauge invariant quantities. a) An ordered string of U's capped by fermions and b) closed Wilson loops.

There are two types of gauge invariant objects one can construct on the lattice: a string capped by a fermion and an antifermion (fig. 2.1a)

$$\overline{\psi}(x)\ U_\mu(x)\ U_\nu(x+\hat{\mu})\ldots U_\rho(y-\hat{\rho})\ \psi(y) \tag{2.7}$$

and the trace of any closed Wilson loop; the simplest example is a plaquette – a 1×1 loop (fig. 2.1b)

$$W_{\mu\nu} \ = \ \frac{1}{N}\ \Re\ Tr\ \left(U_\mu(x)\ U_\nu(x+\hat{\mu})\ U_\mu^\dagger(x+\hat{\nu})\ U_\nu^\dagger(x)\right) \tag{2.8a}$$

but, in general, any closed loop in any representation of color $SU(3)$ is allowed. Here N is the number of colors and $\Re$ is the real part of the trace. The imaginary part can be used to probe electric and magnetic field strengths since it is proportional to $F_{\mu\nu}$. As an exercise convince yourself that there does not exist any other independent gauge invariant quantity.

The lattice action for QCD should be expressed in terms of these two gauge invariant quantities. Consider the simplest string in a form chosen with hindsight

$$\overline{\psi}(x)\gamma_\mu[U_\mu(x)\psi(x+\hat{\mu}) \ - \ U_\mu^\dagger(x-\hat{\mu})\ \psi(x-\hat{\mu})]$$

and expand in powers of the lattice spacing a. Keeping only terms to $O(a)$ one gets

$$= \ \overline{\psi}(x)\gamma_\mu[\ (1+iagA_\mu(x+\frac{\hat{\mu}}{2}))\ \psi(x+\hat{\mu}) \ - \ (1-iagA_\mu(x-\frac{\hat{\mu}}{2}))\ \psi(x-\hat{\mu})\ ]$$

Now Taylor expanding about x

$$= \ \overline{\psi}(x)\gamma_\mu[\psi(x+\hat{\mu})-\psi(x-\hat{\mu})] \ + \ iag\overline{\psi}(x)\gamma_\mu[A_\mu(x)-\frac{a}{2}\partial_\mu A_\mu(x)][\psi(x)-a\partial_\mu\psi(x)]$$
$$+ \ iag\overline{\psi}(x)\gamma_\mu[A_\mu(x)+\frac{a}{2}\partial_\mu A_\mu(x)][\psi(x)+a\partial_\mu\psi(x)]$$

gives to $O(a)$

$$= \ 2a\overline{\psi}(x)\gamma_\mu\partial_\mu\psi(x) \ + \ 2iag\overline{\psi}(x)\gamma_\mu A_\mu\psi(x) \ = \ 2a\overline{\psi}(x)\gamma_\mu[\partial_\mu+igA_\mu]\ \psi(x)$$

This is the standard (Euclidean) continuum fermion action. For the gauge action, I will work through the simple abelian model (in this case the U are commuting complex numbers) and leave the non-abelian case as homework (see section 3 of Ref. [4] for details). The steps and the final functional form (up to numeric factors) are the same for QCD, except that the non-commuting λ matrices make the algebra longer. Lets consider the simplest Wilson loop — the 1×1 plaquette:

$$U_\mu U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x) \ \approx$$
$$\exp[iag\ (A_\mu(x+\frac{\hat{\mu}}{2})+A_\nu(x+\hat{\mu}+\frac{\hat{\nu}}{2}) \ - \ A_\mu(x+\hat{\nu}+\frac{\hat{\mu}}{2}) \ - \ A_\nu(x+\frac{\hat{\nu}}{2}))]$$

Expanding about $y \equiv x + \frac{\mu+\nu}{2}$ gives

$$= \exp\left[ia^2 y(\partial_\mu A_\nu - \partial_\nu A_\mu)\right]$$
$$\approx 1 + ia^2 g F_{\mu\nu} - \frac{a^4 g^2}{2} F_{\mu\nu}F^{\mu\nu} + O(a^6) + \ldots$$

Thus

$$(1 - W_{\mu\nu}) = \frac{a^4 g^2}{2} F_{\mu\nu}F^{\mu\nu} + \textit{terms higher order in } a \qquad (2.8b)$$

since the real part of the linear term is zero. So far the indices μ and ν are uncontracted. There are 6 distinct positively oriented plaquettes, $\mu < \nu$, associated with each site. Summing over μ, ν with $\mu \neq \nu$ and taking care of the double counting by an extra factor of $\frac{1}{2}$, we get

$$\frac{1}{g^2}\sum_x \sum_{\mu\neq\nu}(1 - W_{\mu\nu}) = \frac{a^4}{4}\sum_x \sum_{\mu\neq\nu} F_{\mu\nu}F^{\mu\nu} \rightarrow \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \ .$$

To lowest order in a, we have again recovered the continuum action.

The simplest lattice action for a non-abelian theory is (number of colors $N \geq 2$):

$$A = \frac{2N}{g^2}\sum_x \sum_{\mu\neq\nu}(1 - W_{\mu\nu})$$
$$+ \frac{1}{2}\sum_x \overline{\psi}\gamma_\mu\left[U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})\right]$$
$$+ m_q \sum_x \overline{\psi}(x)\psi(x) \qquad (2.9)$$

which in addition to gauge invariance has the following space-time symmetries:

1) Translations by lattice spacing a.
2) Rotations by 90^o. The rotation group reduces to the discrete hypercubic group.
3) Reflections about sites ($x_\mu \rightarrow -x_\mu$) and about the planes.

These symmetries become the full Poincaré group plus reflections in the continuum limit.

Given the action in eqn.(2.9), the important question is: what are the corrections to observables at finite lattice spacing a? In eqn.(2.8b) we saw that corrections in the gauge part begin at $O(a^2)$ (as a homework assignment show that a general fermion action has corrections starting at $O(a)$). The corresponding operators are called irrelevant because they vanish in the limit $a \rightarrow 0$ but on the lattice they give rise to scaling violations. These can be large if a is not small as is the case in numerical simulations. I will discuss scaling (the necessary condition for getting continuum results from the lattice) in lecture 3.2 and outline an improvement program for pure gauge theory next.

2.3) Improved Gauge Actions

All Wilson loops (of size $i \times j$) have an expansion similar to eqn. (2.8b). The generic form, including the next to leading continuum operators, is

$$\begin{aligned}\frac{1}{a^4}(1 - W(i,j)) &= c_0(i,j)\sum_{\mu\nu} F_{\mu\nu}F_{\mu\nu} \\ &+ a^2\Big[\sum_{\mu\nu} c_2^1(i,j) D_\mu F_{\mu\nu}\, D_\mu F_{\mu\nu} + \sum_{\mu\nu\rho} c_2^2(i,j) D_\mu F_{\nu\rho}\, D_\mu F_{\nu\rho} \\ &+ \sum_{\mu\nu\rho} c_2^3(i,j) D_\mu F_{\mu\rho}\, D_\nu F_{\nu\rho}\Big] + \ldots \qquad (2.10)\end{aligned}$$

Only the numeric coefficients $c_n^\alpha(i,j)$ depend on the particular loop. Since the higher dimension operators are accompanied by powers of a, all lattice actions reduce to the same form in the limit $a \to 0$ (which defines the physical theory). There is no unique lattice action corresponding to a given continuum theory. For $a \neq 0$, the physical quantities calculated with different actions have corrections of $O(a^2)$ with coefficients that depend on the $c(i,j)$. For fermions the corrections begin at $O(a)$. To reduce these corrections we can exploit the freedom to add any number of gauge invariant terms to eqn.(2.9), *i.e.* add arbitrary shape strings to the fermion part and Wilson loops of arbitrary shape, size and representation to the gauge part [9] . Then by a successively more complicated linear combinations of terms in the action we can cancel all $O(a)$, $O(a^2)\ldots$ scaling deviations in observables at finite a. Thereby we have used the arbitrariness of the lattice action into an asset at the expense of a more complicated action which is harder to simulate. The hope is that significant improvement is obtained with a few local terms. Systematic programs to calculate such "improved actions" are being investigated and for details see refs. [10] [11] .

2.4) Fermion Actions and Simulations

Fermions anticommute, so they cannot be simulated directly. The standard procedure is to integrate them out. The formal identity that allows this is

$$\int D\psi\, D\bar{\psi}\, e^{\bar{\psi}M\psi} \qquad det\ M \qquad \int D\phi\, D\phi^*\, e^{-\phi^\dagger M^{-1}\phi} \qquad (2.12)$$

where M $(\not{D} + m)$ is the Dirac operator and ϕ are ordinary numbers called pseudo-fermions. The partition function for QCD in Euclidean space can then be written in a number of equivalent forms:

$$Z \quad \int \mathrm{D}\psi\mathrm{D}\psi^\dagger \mathrm{D}U\, exp(S_G + \psi(\not{D} + m)\psi) \qquad (2.13a)$$

$$Z \quad \int \mathrm{D}U\, det(\not{D} + m)\, exp(S_G) \qquad (2.13b)$$

$$Z \quad \int \mathrm{D}U\mathrm{D}\phi\mathrm{D}\phi^\dagger exp(S_G \qquad \phi^\dagger \frac{1}{\not{D} + m}\phi) \qquad (2.13c)$$

S_G is the gauge action (which for present we shall take to be the simple Wilson action). $\not{D}$ is the fermion covariant derivative, and m the quark mass. Either form (2.13b) or (2.13c) can be simulated, but each requires the calculation of a non-local quantity. So the bottleneck in fermion simulations is that a direct evaluation of either $det(\not{D}+m)$ or $\frac{1}{\not{D}+m}$ is prohibitively slow. The simplest approximation -called the quenched approximation - is to set $det(\not{D}+m)=1$. The gauge configurations then do not include the effects of vacuum polarization. The best interpretation of this approximation is that dynamical quarks have mass $m=\infty$.

The lattice formulation of fermions (definition of operator M) is still not satisfactory. The problem with the naive action in eqn.(2.9) is that in the continuum limit it corresponds to 16 rather than one flavor. In each of the d-dimensions, the propagator has a pole at $p_\mu = 0$ and π *i.e.* at each of the 2^d corners of the Brillouin cell. This can be seen by looking at the inverse of the free field propagator

$$M(p) \;=\; m_q \;+\; i\sum_\mu \gamma_\mu \;\sin p_\mu \tag{2.14}$$

in the limit $m_q \to 0$. This proliferation of flavors is unacceptable. Since the lattice action is not unique, we can play around with the fermion action for cures (for example add irrelevant operators). Before doing that let me first state some important properties we wish ideal lattice fermions to have:

[1] There should be a well defined chiral limit for all g.

[2] For a single Dirac fermion, there should be only one lattice fermion. The spin and flavor degrees of freedom should have one to one correspondence so that continuum operators can be transcribed straightforwardly onto the lattice.

[3] The continuum flavor symmetry in the massless limit is $U_V(1)\times U_A(1)\times SU_V(n_f)\times SU_A(n_f)$. The $U_V(1)$ is baryon number conservation and is not broken. The $U_A(1)$ is broken by instanton contribution and the flavor singlet axial current is anomalous. The $SU(n_f)\times SU(n_f)$ is spontaneously broken to $SU_V(n_f)$ with goldstone bosons (pions) associated with the broken generators. We wish to retain these chiral symmetries in detail.

[4] The fermion action should be local *i.e.* it should couple sites within a small neighborhood only.

Clearly, I would not even be harping on the issue if lattice regularization did not drastically effect the fermions. A theorem by Nielsen and Ninomiya [12] states that for a local fermion action a continuous chiral symmetry is possible only at the expense of fermion species doubling. So far there is no formulation of lattice fermions that preserves chiral symmetry and a one to one correspondance with continuum spin and flavor degrees

of Dirac fermions. There are two popular formulations -- Wilson and Staggered -- that provide a partial fix. Each has its advantages and disadvantages. Here I will only catalogue their main properties.

The fix that Wilson proposed was to add a higher dimension operator which gave fifteen species, $p_\mu \neq 0$, a mass proportional to $\frac{1}{a}$ [13] . These decouple in the continuum limit except for their contribution to the triangle diagram where they act like a Pauli-Villars regularization to produce the correct anomaly. The Wilson fermion (WF) action is

$$M[U]_{i,j} = \delta_{ij} + \kappa \sum_\mu \left[(\gamma_\mu - r) U_{i,\mu} \delta_{i,j-\mu} - (\gamma_\mu + r) U^\dagger_{i-\mu,\mu} \delta_{i,j+\mu} \right] \tag{2.15}$$

where the Euclidean γ matrices are hermitian and satisfy $\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu}$. A common representation is

$$\vec{\gamma} = \begin{pmatrix} 0 & i\vec{\sigma} \\ -i\vec{\sigma} & 0 \end{pmatrix} , \quad \gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.16}$$

Due to the r term, the free propagator

$$M^{-1}(p) = \frac{1}{1 - 2\kappa \sum_\mu \left(r \cos p_\mu - i\gamma_\mu \sin p_\mu \right)} \tag{2.17}$$

no longer has poles at $p_\mu = \pi$. Note that the operator M is not hermitian but satisfies $\gamma_5 M \gamma_5 = M^\dagger$. The same relation holds for M^{-1} also.

Wilson's fix has a drawback; chiral symmetry is explicitly broken. Even defining the quark mass is not straightforward. In the limit $a \to 0$ (free fermion theory), the quark mass is given in terms of the lattice parameters by

$$r m_q = \frac{1}{2\kappa} - \frac{8\kappa r}{2\kappa} \tag{2.18}$$

so we know how to tune it to zero. Away from $a = 0$ we have no a priori knowledge of how to take the zero mass limit. The best we can do is to calculate the pion mass as a function of κ at fixed g, determine (by extrapolation) the value of $\kappa = \kappa_c$ where it goes to zero and call that the zero mass limit. In doing so we have had to explicitly assume the chiral behavior $m_\pi^2 \propto m_q$. Note that this determination of κ_c is statistical. On a given set of configurations, the zero mode occurs for κ in a range about κ_c. So, in addition to the lattice size, this range limits how small a quark mass can be probed at a given g.

Let me give a couple of consequences of the lack of chiral symmetry; 1) if we calculate $\langle \bar{\psi}\psi \rangle$ in perturbation theory, it does not vanish at κ_c. Thus, in order to define the physical $\langle \bar{\psi}\psi \rangle$ we have to subtract the perturbative answer in addition to non-perturbative subtractions. 2) Operators of different chirality mix in the Callen Symanzik equations. This

is true even at κ_c. An example where this is relevant is the calculation of strong interaction corrections to weak interaction matrix elements. For a expòse to the operator mixing problem see Refs. [14] [15] . Further, one needs the spontaneously broken $SU_A(n_f)$, which give rise to certain ward identities, to derive the chiral behavior of matrix elements of the 4-fermi operators. In this respect Staggered fermions (discussed below) are better.

The advantage of WF is that spin and flavor degrees of freedom are identified with continuum Dirac fermions. So it is easy to construct interpolating field operators.

The staggered fermion χ (SF) [16] has only color degrees of freedom at each site *i.e.* no spin. The matrix M is

$$M[U]_{i,j} = m\delta_{ij} + \frac{1}{2}\sum_{\mu} \eta_{i,\mu}\left[U_{i,\mu}\delta_{i,j-\mu} - U^{\dagger}_{i-\mu,\mu}\delta_{i,j+\mu}\right] \tag{2.19}$$

The SF action has translation invariance under shifts by $2a$ due to the phase factors

$$\eta_{i,\mu} \equiv (-1)^{\sum_{\nu<\mu} x_{i,\nu}} \quad . \tag{2.20}$$

Thus a 2^4 hypercube is mapped to a point in the continuum limit. The 16 degrees of freedom in a hypercube become the four spin components of four degenerate flavors. On the lattice, these spin and flavor degrees of freedom are mixed up. Worse, the flavor symmetry is broken and we have no a *priori* way of knowing the value of g at which it is dynamically restored to a desired accuracy (in Monte Carlo calculations this value of g is estimated by requiring that the 15 goldstone modes become degenerate). This mixing of spin and flavor is the major disadvantage of staggered fermions.

SF have a well defined chiral limit at $m = 0$ for all g. This action also has a remnant continuous $U(1)$ chiral symmetry which is sufficient to derive Ward identities like in the continuum [17] . This is the major advantage of staggered fermions over Wilson fermions. So by retaining only part of the chiral symmetry, SF reduce the doubling problem to four.

The operator M is not positive definite (for example in the limit $m_c \to 0$, $M^{\dagger}$ M in eqn.(2.19)). So in numerical simulations we use $M^{\dagger}M$. However, since

$$\int D\phi\, D\phi^{*}\, e^{-\phi^{\dagger}(M^{\dagger}M)^{-1}\phi} = det\ M\ det\ M^{\dagger} \tag{2.21}$$

therefore using $M^{\dagger}M$ leads to a doubling of flavors. For staggered fermions this doubling is removed by noting that

$$\int D\phi_e D\phi_e^{*}\, e^{-\phi_e^{\dagger}(M^{\dagger}M)^{-1}\phi_e} = det\ M \quad det\ M^{\dagger} \tag{2.22}$$

where ϕ_e is defined only on even sites and has no spin degrees of freedom. So one is left with the original 4 fold degeneracy. Thus simulations of SF are most natural with multiples of 4 flavors [18] , while those with WF in multiples of two flavors.

The basic building block of fermionic operators is the quark propagator M^{-1}. For example consider the 2-point correlation function for a π^+. Using WF and making a Wick contraction gives

$$\langle(\bar{d}\gamma_5 u)_\tau\ (\bar{u}\gamma_5 d)_0\rangle \ = \ \langle M^{-1}(\tau,0)\gamma_5 M^{-1}(0,\tau)\gamma_5\rangle$$

where the average is over background gauge configurations. This correlation function is shown in Fig. 2.2 and is easy to evaluate once M is inverted.

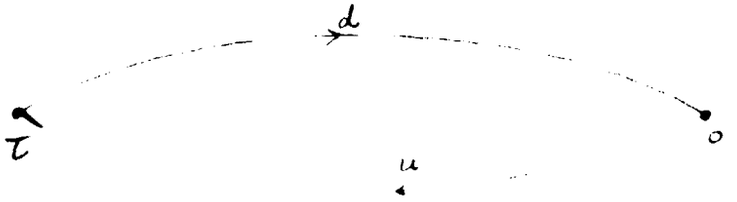

Fig.2.2: The pion correlation function in terms of quark propagators.

A fast algorithm for matrix inversion is crucial to both update and diagnostics. For a comparison of inversion algorithms for WF see [19] . For SF, straightforward conjugate gradient algorithm seems optimal at present. We desperately need an algorithm improvement.

2.5) Numerical Measurement Techniques on the Lattice

There are three ways to do the path integral:

[1] Generate configurations at random, calculate the correlation functions and the weight for the configuration. Then bulk averages are given by

$$\langle O\rangle \ = \ \frac{1}{N}\sum_i^N O_i e^{-S_i} \tag{2.23}$$

with O_i and S_i evaluated on each configuration i. This random sampling method is inefficient and in a finite sample it is easy to miss important configurations all together. It is therefore not used.

[2] Generate configurations by importance sampling i.e. according to the weight e^{-S}. The expectation value is then a simple average:

$$\langle O\rangle \ \ \frac{1}{N}\sum_i^N O_i \tag{2.24}$$

This is the method used in practice and for details see refs. [5] and [20] .

[3] The mixed approach: use the highly fluctuating part S_f of S in an importance sampling algorithm and the smooth part S_s (which is difficult to evaluate) as part of the average.

$$\langle O \rangle = \frac{1}{N} \sum_{i}^{N} O_i e^{-S_s} \tag{2.25}$$

This method has not had any success in QFT because of our lack of ability to do the appropriate factoring. In a different context (here I am getting ahead of myself) for a complex S, which occurs in QCD with finite chemical potential, updating using the absolute value of S and incorporating the phase as part of the observable does not seem to work because the phase has large fluctuations [21] .

At the end of section 2.4, I gave an example of the interpolating operator for the pion and the reduction of its 2-point correlation function in terms of M^{-1}. Similarly, all other fermion correlation functions are constructed from M^{-1} after Wick contractions. Let me now give an example of a pure glue operator for the 0^{++} glueball mass:

$$\mathcal{O}(\tau) = \sum_{x_1, x_2, x_3, i<j} W_{i,j}(x_1, x_2, x_3, \tau) \tag{2.26}$$

where W is any Wilson loop. Recall from eqn.(2.10) that all Wilson loops have an overlap with $F_{\mu\nu}F_{\mu\nu}$. The sum over sites on a time slice τ projects $\mathcal{O}$ onto zero-momentum and the sum over spatial orientations $i < j$ makes it a scalar (in general a multiple of spin 4). The higher spin contamination, we hope, is suppressed by a large mass-gap between these states.

I have already shown that correlation functions correspond to expectation values of time ordered products of operators. Let me now show how mass-gaps (proton, rho, pi *etc.* masses) are measured. Consider the amplitude

$$\langle s|\mathcal{O}(\tau)\mathcal{O}(0)|s\rangle = \langle s|\cdots|s\rangle\langle s|\mathcal{O}(\tau)|{}^{s-1}_{s+1}\rangle\cdots\langle {}^{s}_{s+1}|\mathcal{O}(0)|s\rangle \tag{2.27}$$

where $\mathcal{O}$ is the interpolating field operator and $|s\rangle$ are the eigenstates of the Transfer matrix. Then $\mathcal{O}(0)$ can either create or annihilate a quanta at time 0 while $\mathcal{O}(\tau)$ does the reverse. Remembering that the time evolution of the states $\sim e^{-E_s\tau}$, the two dominant contributions to eqn (2.27) in the limit of large τ are

$$\langle 0|\mathcal{O}(\tau)\mathcal{O}(0)|0\rangle \sim e^{-E\tau}$$
$$\langle 1|\mathcal{O}(\tau)\mathcal{O}(0)|1\rangle \sim e^{-E(L-\tau)}$$

where L is the size of the lattice and E is the mass-gap. The lattice has periodic boundary conditions and the energy of $|0\rangle$ is set to zero. Thus the behavior of the 2-*point* correlation

function in the large τ limit is

$$\begin{aligned} \langle O(\tau)O(0)\rangle &\sim e^{-E\tau} + e^{-E(L-\tau)} \\ &\sim \cosh E(\frac{L}{2} - \tau) \end{aligned} \tag{2.28}$$

So if the data shows a $\cosh E(\frac{L}{2} - \tau)$ behavior for large L and τ, then E can be extracted from the fit.

For a reliable estimate of the mass of a given state we need to do the following:

(a) The operator O should create the state at zero momentum. This is usually accomplished by making the operator translation invariant *i.e.* $O = \sum_x O(x)$.

(b) Choose an operator that couples to a single state. This is in principle hard if not impossible. For example lattice operators that couple to a pion also couple to all radial excitations as well. So the attempt is to maximize the overlap with the lowest state. This can be done by making O a variational wave-function. The method then provides information about the wave-function of the state once the variational parameters are tuned.

(c) Make L and τ large so that even if there is a contamination in O from higher states, the corrections are negligible due to the exponential suppression of higher states at large τ.

We will discuss how well these points are under control case by case later.

2.6) Finite Temperature QFT

The connection between time of propagation of a state in Euclidean QFT and temperature in statistical mechanics is very profound. The quantum eigenstates have a time evolution $\exp(-E_i\tau)$, so the relative contribution of higher energy states is $\exp(-(E_i - E_0)\tau)$ (they are not damped out for finite time propagation). Thus each state in the sum over states has the correct Boltzmann like factor to allow an interpretation of τ as inverse temperature. In short, a system in a finite box in the "time" direction is at finite temperature. This is the basic idea of finite temperature QFT (see Ref. [22] for a review).

To do QCD calculations at finite temperature, we label one direction as time and take a finite number N_t of lattice points along it. The temperature T of the system is then given by $T = \frac{1}{N_t a}$ where a is fixed by g. The boundary conditions in the time direction should be periodic (anti) for bosons (fermions). In the spatial direction the boundary conditions can be chosen to be whatever mimics infinite volume best. To vary the temperature, one fixes N_t and changes g continuously to change the scale a. The continuum behavior is obtained when N_t is large (*i.e.* a is small enough so that scaling has set in) and the spatial volume is "infinite".

To calculate the zero temperature behavior of the theory, the smallest lattice dimension N has to be large enough so that $e^{-(E_1-E_0)N}$ is negligible. In practice lattices are finite, so all physical quantities are measured at small but finite temperature. The $T = 0$ limit is gotten by extrapolation. For a sensible extrapolation there are two prerequisites: 1) There should be no singularities (finite temperature phase transitions) as N is increased and 2) there exist some functional form to match on to at large N.

3) SCALING, CONTINUUM LIMIT AND EFFECTIVE THEORIES

A fundamental QFT is one for which the limit $a \to 0$ can be taken self-consistently. In this limit the theory is non-trivial (interacting) if the renormalized couplings g_R are non-zero and finite. The lattice serves simply as a regulator. To extract physical numbers from a given lattice theory we need to know 1) the g_{bare} at which to take $a \to 0$ and 2) how this limit is approached so that one can extract sensible numbers working at finite a. In this lecture I will discuss these points with simple physical observables like hadron masses as examples.

As discussed in section 2.5, the mass m of an excitation of the theory (states) is measured from the exponential fall-off of the 2-point correlation function. The correlation length (measured in dimensionless lattice units) is

$$\xi = \frac{1}{m(a)a} \tag{3.1}$$

where m is expressed in GeV, and a in GeV^{-1}. The mass of the physical state is given by

$$m = \lim_{a \to 0} m(a) \tag{3.2}$$

which remains finite as $a \to 0$ for a fundamental theory. The continuum theory is defined at points where $\xi \to \infty$. As in statistical mechanics, points at which correlation lengths diverge are called critical points. It is easy to see that at such points the correlation lengths of all physical excitations $\to \infty$. A finite correlation length ξ is possible only if $m(a) \to \infty$ as $a \to 0$. Such states decouple from the theory.

So to have a fundamental QFT that describes nature we require the following:

(a) There exist at least one non-trivial fixed point of the theory. A fixed point is a special critical point at which the theory becomes scale invariant.

(b) In case the theory possesses many distinct fixed points, the relevant one is that about which the mass-ratios agree with nature.

(c) Universality: The fixed point governs the physics (mass-ratios) in a certain neighborhood about it. Said another way, all actions (in the infinite dimensional space of coupling constants) about the fixed point yield the same long distance behavior with corrections that vanish as powers of a. Thus we can make simulations away from $\xi = \infty$ and successively decrease the error in a controlled fashion. For QCD, our optimistic hope (buffered by some numerical evidence) is that this region extends to $g_{bare} \approx 1$ where the correlation length is ≈ 10. So realistic calculations are feasible.

In general, a theory is non-perturbative about its non-trivial fixed points. There is one notable exception -- asymptotically free field theories (AFFT). They have a non-trivial ultraviolet stable fixed point at $g_{bare} = 0$. The point $g_{bare} = 0$ is always a fixed point, but it is usually a trivial fixed point because the theory reduces to a gaussian model at this point ($g = 0 \Rightarrow$ no interactions). What distinguishes an AFFT from a trivial gaussian model is that for AFFT, g_R tends to a nonzero constant as $g_{bare} \to 0$. This completes our first goal which was to identify the relevant fixed point.

The advantage of AFFT is obvious: Near this fixed point perturbation theory is valid. Thus one can calculate how n-point functions change under a scale change. This evolution is described by the Callen-Symanzik equations. The simplest of these is for the coupling constant g

$$a\frac{\partial g(a)}{\partial a} = -\beta(g(a)) \tag{3.3}$$

where $\beta(g)$ is called the β-function. For non-abelian theories, perturbation theory gives

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \cdots \tag{3.4}$$

Only the first two coefficients, β_0 and β_1 are regularization scheme independent. They are

$$\beta_0 = \frac{11N - 2n_f}{3 * 16\pi^2} \tag{3.5a}$$

$$\beta_1 = \frac{34N^2 - 10Nn_f - 3n_f(N^2-1)/N}{3 * (16\pi^2)^2} \tag{3.5b}$$

where N is the number of colors and n_f the number of quark flavors. Thus, the $\beta-$function is negative for the anticipated 6 flavors in the standard model. Because of this important property, $g \to 0$ as $a \to 0$ and the fixed point at $g_{bare} = 0$ is ultraviolet stable (for details see lectures by J. Rosner in this volume).

At $g_{bare} = 0$, the correlation lengths diverge. Lattice simulations have to extrapolate from finite a. This extrapolation is called taking the continuum limit. By consistency, we take the limit $a \to 0$ holding some physical quantity constant (the proton mass for example). Let $m_L = ma$ be the dimensionless mass measured on the lattice. Then

$$\frac{dm_L}{dg} = m\frac{da}{dg} . \tag{3.6}$$

Holding m fixed prescribes how a varies as $g \rightarrow$. So when one measures m_L at a number of values of g, it changes because a (an arbitrary unit) changes. Such measurements give a non-perturbative answer for the relation between a and g. But how do we know it is right and how do we extract physical masses?

Integrating the 2−loop β−function, eqn (3.3), tells us how any dimensionless mass (measured in lattice units) should change with g for small g

$$ma = cf(g) \equiv c\,(\frac{1}{\beta_0 g^2})^{-\frac{\beta_1}{2\beta_0^2}}\, exp[-\frac{1}{2\beta_0 g^2}] \tag{3.7}$$

Eqn.(3.7) holds for all physical observables with dimensions of mass. Only the constant c is different. So once the mass measured on the lattice starts to change according to eqn.(3.7), eqn. (3 3) and (3.6) agree up to some predefined error criterion, then that interval $[g_{asy}, 0]$ is called the asymptotic scaling region. The constants c_i are extracted from a best fit to $m_L/f(g)$.

Having gotten the c_i, we still do not have a physical mass since $f(g)$ is arbitrary at least up to a multiplicative factor. However, their ratios are well defined and specify the spectrum of the theory. If these numbers agree with experimental values then we have the right theory.

Another approach is to define an integration constant $\Lambda \equiv \frac{f(g)}{a}$ with $f(g)$ given by eqn (3.7). Then $m_i = c_i\Lambda$ in the asymptotic scaling region and all dependence on g_{bare} and the cutoff a is hidden in Λ. This integration constant provides a scale and corresponds to defining g_R, the one unknown parameter of the theory. QCD, ignoring quark masses, has only one dynamically generated scale. To fix Λ in physical units (MeV), one relates it to a continuum definition like $\Lambda_{\overline{MS}}$ by one loop perturbation theory and then uses the experimental value for $\Lambda_{\overline{MS}}$. Of course by reversing the argument one could predict $\Lambda_{\overline{MS}}$ provided all calculations are done at small enough g so that perturbation theory is valid.

The replacement of the coupling constant by a momentum scale Λ is given a fancy name −− dimensional transmutation. The idea is neat; we started with a scale invariant theory (ignoring quark masses) and find that in order to get sensible results all quantities have to be defined with respect to some momentum cutoff. One can change the cutoff and physical quantities like masses remain unchanged if we suitable adjust the running coupling constant (for n−point functions we have to include anomalous dimension factors). This renormalization group invariance is embodied in eqn.(3.7).

For $g \approx 1$, we have no a priori reason to believe that eqn.(3.7) works. However, we can still write

$$m_i a \qquad c_i f_i(g) \tag{3.8}$$

where $f_i(g)$ are unknown non-perturbative functions which reduce to eqn (3.7) in the limit

$g \to 0$. The expected behavior of mass ratios in QCD is

$$\frac{m_i}{m_j} = \frac{c_i}{c_j}(1 + O(a^2 \ln a)) \tag{3.9}$$

and it is believed that corrections at $g \leq 1$ to the ratios (coefficient of the $O(a^2 \ln a)$ term) are small compared to those for individual masses. The interval $[g_s, 0]$ over which mass ratios are constant is called the scaling region. We do know that $g_s \geq g_{asy}$. The present numerical data does not provide an unequivocal answer for g_s. So achieving constant mass ratios remains the goal of lattice calculations.

Let me restate these ideas from an operational stand-point. A lattice simulation is done on a grid of size L^4 with two bare parameters g and $m_q a$ (assuming flavor $SU(2)$ is unbroken so that $m_q = m_u = m_d$). We then measure pion and rho masses, which in lattice units are pure numbers *i.e.* $m_i(a)a = c_i$. By fiat we declare $m_\rho = 770\ MeV$. Then

$$\frac{1}{a} = \frac{m_\rho}{c_\rho} = \frac{770}{c_\rho}\ MeV \tag{3.10}$$

with c_ρ evaluated at that $m_q a$ for which $\frac{m_\rho}{m_\pi} = \frac{c_\rho}{c_\pi} = \frac{770}{140}$. So we have used two known masses to fix the two bare parameters (remember g and a are related by eqn.(3.7)). Now, the test that QCD is the correct theory is that all other masses calculated – proton, delta baryon, glueballs – agree with nature.

In the above procedure the value for g was not specified. So you could ask whether there exists another fixed point (different from $g_{bare} = 0$) at which all mass ratios take on physical values. If QCD is the fundamental underlying theory, then the answer is no. All we want to know is how small g has to be for reliable results. For $g < g_s$, as $g \to 0$, all mass ratios become constant with corrections that vanish exponentially with g. Moreover the variation of any mass measured in lattice units begins to show a g dependence prescribed by the 2-loop result eqn.(3.7) for $g < g_{asm}$. This as mentioned earlier is a consequence of universality about the UV fixed point. So a good test of numerical calculations is that scaling behavior is demonstrated over a scale change of two or more.

To summarize, the lattice theory is trivial at $a = g = 0$ and all $\xi = \infty$; physics is extracted from the way physical quantities change as $a \to 0$. At finite a all correlation lengths are finite. The infinity arises because we start to measure a finite distance with an increasingly fine scale. Introduction of a unit, Λ, gets rid of the spurious singularity. However, the increasingly fine scale is necessary to remove errors due to the discrete lattice formulation. As $g \to 0$, the different correlation lengths (corresponding to different physical excitations) do not diverge independently. They can all be expressed as a constant times a given selected one. This is what one means by "the theory has a single mass/length scale".

These previous ideas of the continuum limit and scaling are forged by Wilson's formulation of the renormalization group. It is a central concept in modern field theory and you are encouraged to work through Ref. [23] .

In statistical mechanics the intramolecular separation (call it a again) is the natural unit for lattice spacing while temperature and magnetic field are physical measureable quantities. These couplings of the "effective" theory are in principle derivable from a fundamental microscopic theory, but for understanding macroscopic phenomenon we are not interested in that connection. Starting with an effective Hamiltonian defined at scale a, a lattice simulation describes physics for all $r > a$. By varying the couplings one can investigate the behavior at different temperature, magnetic field *etc.*

In SM, models representing nature have to reproduce phenomenon at all physical distance scales and over a range of parameters (like temperature *etc*). To do this faithfully, we need an infinitely complicated effective theory. Fortunately, simple models can describe the behavior in limited domains. Of particular interest are critical points at which substances undergo a drastic change in behavior. They show collective behavior with fluctuations at all length scales. The 2-point correlation function no longer fall off exponentially but with a power-law. This is characterized by $\xi \to \infty$. One would have thought that such regions are complicated to analyze. The simplicity arises because the correlation functions (n-point Green's functions) scale, *i.e.* are dominated by a single large correlation length. The many-variable functions become simple functions of a few reduced variables and scale changes are expressable by simple equations (the Callen-Symanzik equations). These equations simplify further about the fixed point. The solutions to these idealized equations are excellent guides to what one really observes in nature for distance scales much larger than the granularity. This is the scaling hypotheses [24] .

One talks of critical points in both QFT and SM, but there is a subtle difference that should be noted. In QFT, critical points are simply points at which the lattice scaffolding can be discarded. So, tuning ($a \to 0$) is necessary to get the continuum theory.

There is often confusion about critical points and fixed points. Fixed points are special critical points at which the theory becomes exactly scale invariant. At critical point there are corrections that go under the name of scaling violations. A generic form of the correction is $\frac{f(r)}{r}$ where $f(r)$ is a slowly varying function of r as $\xi \to \infty$. Thus for large distance behavior *i.e.* $r \approx \xi$ as $\xi \to \infty$, the distinction between critical and fixed point fades. It is worth remembering that physical systems exhibit critical behavior. Fixed points are mathematical constructs.

Now lets turn to effective/phenomenological theories in QFT. They differ from fundamental theories in that they are valid only up to some short distance cut-off. For energies much smaller than the cutoff, the presence of the cut-off is a mild curiosity and effective theories are an accurate description of nature. They have an advantage of focusing attention on the relevant degrees of freedom at that scale. Only for energies close to the cut-off, the need for an underlying fundamental theory (or next level of effective theory) becomes manifest. Effective theories can be derived from a microscopic theory by integrating out

all degrees of freedom with momentum larger than a predefined cutoff. More often they arise de facto; a theory is invented that describes phenomenon at a given scale and later it is shown to be valid only up to some finite cutoff. An example is the 4-fermi theory of weak interactions. In the next section I describe another possible relevant example of an effective theory -- electroweak interactions in the standard model.

3.1) The $SU(2) \times U(1)_Y$ + *Higgs* model

The bosonic sector of electroweak interactions is described by the spontaneously broken gauge group $SU(2) \times U(1)_Y$ coupled to a doublet of higgs in the fundamental representation of $SU(2)$. So far no non-trivial fixed point is known for either the higgs model or the $U(1)$ theory. It is not known if coupling to $SU(2)$ and the occurrence of spontaneous symmetry breaking gives rise to a non-trivial fixed point. While the presence/absence of a non-trivial fixed point is an open problem, it is fairly certain that if such a fixed point exists then it would be inherently non-perturbative so lattice calculations are our best bet for settling the issue.

In the absence of a non-trivial fixed point the electroweak theory is a low energy effective theory. Let me first assume that this is the case. We would then like to know under what circumstances is the Higgs particle mass predictable or how to derive an upper bound on it. Let me briefly outline how the community is going about addressing this question. For details see the reviews by Shrock [25] and Jersak [26] .

A standard form for coupling the gauge interactions to Higgs is to modify the derivative term in eqn.(2.1). The full action is $S_H + S_1 + S_2$ with

$$S_H = \sum_x \Big[-\kappa \Re(\phi^\dagger(x) \sum_{i\,\mu} U^2_\mu U^1_\mu \phi(x+\hat{\mu})) + |\phi(x)|^2 + \lambda(|\phi(x)|^2 - 1)^2 \Big] \quad (3.11a)$$

$$S_1 = \frac{1}{g_1^2} \sum_x Tr(1 - U^1_p) \quad (3.11b)$$

$$S_2 = \frac{4}{g_2^2} \sum_x \frac{1}{2} Tr(1 - U^2_p) \quad (3.11c)$$

where U^2_μ (U^2_p) and U^1_μ (U^1_p) are link (plaquette) variables of the $SU(2)$ and $U(1)$ theory respectively and ϕ is a complex doublet of fields. The theory has four independent couplings g_2, g_1, λ and κ. The scalar action in the continuum is

$$S_H = \int d^4x \left(\frac{1}{2} \varphi^\dagger(x) D_\mu D^\mu \varphi(x) + \frac{m^2}{2} \varphi^\dagger \varphi + \frac{\lambda}{4} (\varphi^\dagger \varphi)^2 \right) \quad (3.12)$$

and the relation between lattice and continuum quantities is given in equations (2.2). In the following I shall use φ (ϕ) to denote the continuum (lattice) field.

In the limit $g_2 = g_1 = 0$, the Higgs model has an $O(4)$ global symmetry with a line of critical points given by $\kappa_c(\lambda)$ [23]. For small λ one gets a decent approximation from eqn. (2.2b) *i.e.* $\kappa_c(\lambda) \approx \frac{1-2\lambda}{8}$. There is mounting evidence that the continuum limit of this theory taken anywhere along this line is trivial *i.e.* the model reduces to a gaussian one For $\kappa > \kappa_c$, the global $O(4)$ is spontaneously broken and the spectrum consists of three goldstone bosons and a massive Higgs. On the lattice, one can measure two quantities m_s and $\langle\phi\rangle$ to fix the bare couplings and the wavefunction renormalization Z_ϕ to relate them to the renormalized quantities. For $\kappa \to \kappa_c$, the predictions for the $O(N)$ gaussian model are [27]

$$m_s^2 \propto \kappa_r (\log \kappa_r)^{-\frac{n+2}{n+8}} \tag{3.13a}$$

$$\langle\phi^2\rangle \propto \kappa_r (\log \kappa_r)^{\frac{6}{n+8}} \tag{3.13b}$$

$$\frac{m_s^2}{\langle\phi^2\rangle} \propto (\log \kappa_r)^{-1} \tag{3.13c}$$

where $\kappa_r \equiv |\kappa - \kappa_c|$. Eqn. (3.13c) tells us that the Higgs mass goes to zero at all points along the critical line. On the other hand, when $m_s a \approx 1$, a cut-off theory ceases to make sense and estimates of the Higgs mass become unreliable. In between these two limit cases one has a sensible effective theory with $\lambda_R \neq 0$. The renormalized coupling and field strength can be defined to be

$$\lambda_R \equiv \frac{m_s^2}{\langle\phi_R^2\rangle} \tag{3.14a}$$

$$\langle\phi^2\rangle = Z_\phi \langle\phi_R^2\rangle \tag{3.14b}$$

where the Z_ϕ factor depends on the renormalization scheme and is different on the lattice and in the continuum. All renormalized quantities are henceforth denoted with a subscript R.

To predict the Higgs mass we need to relate $\langle\phi_R\rangle$ to M_W [28] . This is done by adding the $SU(2)$ gauge interactions by hand using the tree-level continuum relations

$$M_W^2 \quad \frac{g_R^2}{4} \langle\varphi_R^2\rangle \tag{3.15a}$$

$$\frac{M_H^2}{M_W^2} \quad \frac{4\lambda_R}{g_R^2} \tag{3.15b}$$

$$g_R^2 \quad 4\sqrt{2} M_W^2 G_F \approx 0.4 \tag{3.15c}$$

where we put $m_s \quad M_H$ and use $G_F \quad 1.166 \times 10^{-5} GeV^{-2}$. Note that in the absence of $U(1)$ there is no weak mixing angle. Since λ_R increases as the cut-off is decreased, the upper bound on eqn. (3.15b) is at the smallest cut-off (of course larger than M_H) for which scaling occurs.

Two different numerical approaches have been used to measure m_s, $\langle\phi^2\rangle$ and Z_ϕ on the lattice. For a description and details see Refs. [29] [30] and [31] [32] For a fixed cutoff (*i.e.* a fixed value of $m_s a$) the largest upper bound on the Higgs mass occurs for $\lambda \to \infty$. The present estimate from these calculations is $\frac{m_H}{m_W} \approx 10$ evaluated at $m_s a \approx 0.2$. In the last year, Lüscher and Weisz have developed analytic methods (combining renormalization group equations with high temperature series expansion) to analyse the $\lambda\phi^4$ model and calculate the bound on the Higgs mass [33] . These techniques are well worth learning and I encourage you to work through these papers.

The next step is to include the $SU(2)$ gauge interactions in lattice simulations. The qualitative change that occurs is that in the Higgs phase the goldstone bosons of the $O(4)$ theory become the longitudinal excitations of the gauge bosons and this combination gives the three massive W-bosons. Other than that, for small g_2, we assume that the gauge interactions can be handled perturbatively and that the physics is controlled by the $O(4)$ fixed point of the Higgs model. Then for fixed λ and g_2, we can calculate m_H and m_W on the lattice as a function of κ in the broken phase near κ_c. To test the assumption of the relevance of the $O(4)$ fixed point, one compares the scaling behavior of the masses against eqn. (3.13) and against eqn. (3.15a). Next vary g_2 and repeat the calculation. Then for a fixed cut-off (*i.e.* $M_W a = constant$), the ratio $\frac{m_H}{m_W}$ is determined as a function of g_2. If the ratio does not change dramatically as a function of g_2 and both m_H and m_W are small compared to the cutoff, then we get a reliable estimate of the ratio. To make a prediction we have to tune g_1 such that $g_R \approx 0.4$ as given by eqn. (3.15c). Again, the upper bound comes from taking $\lambda \to \infty$ for a fixed cut-off and then decreasing the cut-off while staying in the scaling region. If this is the correct scenario, then in addition to a bound on the Higgs mass we have an estimate of the scale (*i.e.* the cut-off) at which new physics sets in.

In the above discussion I have made the implicit assumption that adding $U(1)_Y$ and fermions to $SU(2)$ *Higgs* model do not change the bound very much. The effects of the $U(1)_Y$ can be estimated by incorporating it in lattice simulations and checking the tree level relations involving the weak mixing angle. The fermion sector is an open problem at present since we do not know how to put chiral fermions on the lattice. For some recent work with non-chiral fermions see the review by Shrock.

If we find a non-trivial fixed point in the bosonic sector of the full theory then we first need to show that it is relevant to the standard model by measuring m_s and $\langle\phi\rangle$ and Z_ϕ (or equivalently m_W and m_H) on the lattice. If these agree with nature, then the previous discussion changes only in the following way: the value of lattice couplings at which to make contact with the continuum are determined *i.e.* the mass ratio is calculated as a limiting process by tuning the couplings to their critical point values. There remains only one length scale in the problem, say $\langle\phi\rangle$, and the Higgs mass can be predicted.

The presence of a non trivial fixed point at large g_2 and g_1 is not ruled out by the

known small values of G_F and α_{em} if the model has large renormalization effects. The relation between the bare and renormalized quantities has to be determined by measuring physical observables; for example the potential between two external charges can be used to define g_R in terms of g and eqn.(3.14a) for λ_R. So far there is no evidence for large renormalizations in the broken phase of either the $SU(2)$ *Higgs* model or the $U(1)$ *Higgs* model. Thus the present consensus seems to be that a search for a non-trivial fixed point is a high pay off but a very high risk undertaking. However, since the presence/absence of non-trivial fixed points can only be settled by non-perturbative methods, it is necessary to develop the tools.

4) STATUS OF LATTICE CALCULATIONS

4.1) Necessary Ingredients For A Good Calculation

To make progress while waiting for faster computers, we have to develop a fast update algorithm which includes a fast matrix inversion routine for the quark propagator. To enhance the signal in measurements we need to design better operators. To answer the question — how much better, we have to recognize our goal in 4 categories:

(a) Statistics: Need to develop ever faster algorithms for producing decorrelated configurations. The average over configurations should give a faithful representation of the path integral. Also, one should always explore (and be aware of) variance reduction techniques [34] to reduce errors.

(b) Extrapolation to $m_q a \approx 0$: We need to do the calculations as close to the physical light quark masses as possible. The pion correlation length grows as $\frac{1}{\sqrt{m_q a}}$ so the lattice size has to be adjusted accordingly. The matrix inversion problem suffers from critical slowing down in the limit $m_q a \to 0$. A method that in principal counters this critical slowing down is fourier acceleration. In tests done so far, fourier acceleration gave no significant improvement for Staggered fermions on lattices up to $16^3 \times 32$ [35] . For Wilson fermions, the gain is ≈ 3 and the method seems more promising. We need alternative methods which are significantly faster.

(c) Control Over Finite Volume Effects: The mass estimates should be made close to the infinite volume limit or we must develop demonstrably reliable finite volume extrapolation formulas. Right now we are struggling to make the lattice volume large enough so that the states are not squeezed into a box smaller than their size. A desirable goal is to have a physical lattice size that is at least 4 fermi across. The present box sizes are only ≈ 1.4 fermi.

(d) Extrapolation to $q \to 0$: The lattice introduces scaling violations at finite q that vanish

in the limit $g \to 0$. A reasonable test that the lattice results mimic the continuum is constancy of mass ratios over a range of coupling for which the scale changes by a factor of 2. Finding improved actions, as discussed in section 2.3, to reduce these corrections for some fixed correlation length would help greatly.

A rude estimate of the kind of power we need with present algorithms to perform the next generation quenched calculations is 10^4 gigaflop hours. With dedicated computers being built this goal is within sight. But the need for algorithm development is obvious.

Status of Numerical Results

I will just summarize the key issues and the status of present calculations. For details, see the lectures by M. Fukugita and R. Gupta in [6] and the reviews [36] [37] . A word on notation; the $SU(3)$ gauge coupling is specified in terms of either g or $\beta \equiv \frac{6}{g^2}$.

4.2) The Hadron Spectrum:

The calculation of low lying meson and baryon masses from first principles will be proof that QCD is the correct theory of strong interactions. Thus, many calculations have been done (see Table VII by M. Fukugita [37]). Most large lattice calculations have been done in the quenched approximation. This approximation can at best be justified up to 10% for hadron masses. The true accuracy of the approximation will be known only a *posteriori* after all other sources of error (see section 4.1) are removed.

Present results show a consistently large value for the ratio $\frac{M_N}{M_\rho}$ i.e. > 1.5. The value 1.5 is expected in the limit of infinitely heavy quarks. Since all present calculations the lightest quark mass used is still too large ($m_\rho < 2m_\pi$) we have to resort to extrapolation to get physical mass-ratios. One set of calculations [38] [39] [40] do not give a reasonable estimate for the ratio when results are extrapolated to the physical quark mass. On the other hand a number of calculations; [41] , [42] , [43] [44] do find a smooth extrapolation in m_q and the results are consistent with the physical numbers. Does the difference lie in the fitting procedure to extract the lowest mass and in the final extrapolation in m_q? It would be useful if a single unbiased person analysed all the data starting with the 2 point correlation functions averaged over configurations.

I feel that we need to avoid the extrapolation from large m_q. To go to smaller m_q requires larger lattices (to control finite size effects) and significantly more statistics to overcome the deteriorating signal to noise ratio. The next generation computers will take us closer to the brute force solution — increase the lattice size and the statistics and push towards physical m_q without algorithm improvement.

The calculations with dynamical fermions have been done on smaller lattices and at stronger coupling. More important, they have been done at still heavier quark masses,

i.e. ($m_\rho < 2m_\pi$). The resulting spectrum is essentially the same as the quenched case [45] [37]. [46] [47] [48] This result is not unexpected for heavy quark masses. In fact it is consistent with data for mesons and baryons with all strange quarks.

An additional problem in present lattice calculations is that the baryon mass is large in lattice units *i.e.* $ma \approx 1$. Thus there is no reason to expect that the lattice granularity does not significantly effect its measurements. This possible source of error will also be exposed when the quark mass is decreased.

To sort out the nucleon to meson mass ratio discrepancy and verify things like the nucleon to Delta mass splitting one needs to follow a three step program: 1) For fixed coupling (*say* $\frac{6}{g^2} = 6.0$) make a finite volume study (on lattice sizes $16, 24, 32 \ldots$) varying m_q until one reaches close to the physical quark mass. The purpose here is to eliminate finite size effects and the extrapolation from large m_q. 2) To decrease g until mass ratios at physical quark masses become constant. This eliminates possible scaling violations. 3) Having gained control over statistical and systematic errors repeat the calculation with dynamical fermions.

The positive aspects of quenched calculations are: 1) the spectrum with Wilson and staggered fermions show agreement, starting at $\beta \approx 6.2$ [38]. 2) Similarly, starting at this coupling the staggered flavor symmetry is dynamically restored to $SU(n_f)$ (up to 5%) [38] [39]. 3) In all calculations, the pion shows the expected chiral behavior ($m_\pi^2 \propto m_q$) up to unexpectedly large m_q. It is not clear if this is good or bad considering the $\frac{M_N}{M_\rho}$ problem. Are we being misled by an apparent linear behavior while the true chiral behavior will set-in for much smaller m_q or are the extrapolations for the baryons from heavy m_q at fault? The next generation calculations will address these questions.

4.3) The Heavy $q\bar{q}$ Potential:

Our goal is to predict the toponium spectrum from a lattice derived potential. This lculation relies on our ability to calculate the expectation value of large Wilson loops or 2-point correlation of Wilson lines at large separation. Present estimate of the accuracy necessary in Monte Carlo calculations is $\sim$ 0.5% error in 10×10 loop at $\frac{6}{g^2} = 6.0$. This accuracy factor depends on the string tension (σa^2 is falling exponentially with g) and the size of the loop we need to measure to probe the same physical distance (r/a is increasing exponentially). Based on present estimates of lattice scale a, to extract just the spin independent potential in the range of charmonium to toponium (0.02 fermi to 1 fermi) we need to measure at least $10 \sim 10$ loops all the way from $\frac{6}{g^2} = 6.0$ *to* 7.5. Thus, without an algorithm improvement we are at least a factor of 10^2 short of the computer power needed to provide a reliable answer.

The successes so far are qualitative 1) The shape of the spin independent potential is roughly consistent with the Cornell potential with a linear piece dominating for $r \sim 0.5$

fermi. The small r behavior does not show good agreement and it is speculated that the quenched approximation is at fault. Present calculations with fermions are not accurate enough to shed any new light on this problem. 2) The spin-orbit term V_{ls} in the spin-dependent potential is long range [49] .

For hard numbers, the string tension is the best measured quantity. The world data (with my biased selection) is shown in Table 4.1 [50] [51] [52] [53] [54] [55] [56] . The results suffer from two systematic effects; 1) the difference between Wilson string tension and 't Hooft string tension and 2) it has not been reliably demonstrated that the estimate of σ is independent of loop size (or the separation in the 2-point correlation functions). Thus there may still be large errors in the extraction of σ — $\approx 50\%$ even at $\frac{6}{g^2} \approx 6.0$.

$\frac{6}{g^2}$	σ_W	σ_W	σ_t	σ_t
5.5				0.340(15) [52]
5.6	0.279(9) [50]			
5.7	0.195(10) [53]			0.135(5) [52]
5.8	0.111(3) [50]	0.099(1) [51]		
5.9			0.068(3) [56]	0.062(3) [52] [55]
6.0	0.061(2) [50]	0.046 [54]	0.050(2) [56]	0.042(3) [52]
6.1		0.046* [51]		
6.2	0.036* [50]		0.026(3) [56]	
6.3		0.0173 [54]		

*Table 4.1 : Estimates of the Wilson (σ_W) and 't Hooft (σ_t) string tension. I have included the $\frac{\pi}{3L^2}$ finite volume correction in σ_t. The * against values indicates that the estimate is not asymptotic. Note the systematic error when more than one group has extracted σ at the same coupling, and also the difference between σ_W and σ_t.*

Calculations with dynamical fermions are still qualitative because small lattices have been used in the region $g \sim g_s$. One does see evidence of screening; the lattice potential does not have as strong a linear rising piece as in pure $SU(3)$ [57] . This is the expected behavior: light quarks will cause the string to break and form mesons when we try to separate a $q\bar{q}$ pair beyond a certain distance.

4.4) The Glueball Spectrum:

The masses and properties of glueballs are a major untested prediction of QCD. For a status report on the experimental status see review by Chanowitz [58] . Since mass calculations are intrinsically non perturbative, the only reliable method is numerical simulations on the lattice. Lattice glueball mass measurements have mainly been done for

the pure gauge theory (preliminary calculations with dynamical fermions have used a very large quark mass). Therefore we have no new information on mixing between glue states and states with the same quantum numbers but with quarks in them.

The two main bottlenecks in glueball calculations have been 1) a rapid loss of signal with increasing β at a given physical separation τ (see eqn.(2.28)). Worse this distance is not large enough to allow a clean single exponential fit to extract the mass of lowest state. 2) large finite volume effects. To solve the first problem we need to invent better interpolating field operators. Because of these problems, the only pure glue state for which we have a modestly reliable estimate is the 0^{++}. By Murphy's law, the experimental information on this state is poor.

The world data from large lattices and for $\frac{6}{g^2} \leq 6.0$ is shown in Table 4.2 along with the string tension result from the same data. The present estimate for the ratio $\frac{M_{0^{++}}}{\sqrt{\sigma}}$ is $3.1 - 3.4(1300 - 1400 MeV)$ [59] [60] [61] [55] [56]. So far this ratio has shown stability under an increase of the lattice volume (except for set 15 in Table 4.2). I feel that we still need more careful checks at a larger lattice volume and larger β.

For the initiated only: I would like to contrast the results of Teper *et.al.* [56] obtained with improved interpolating field operators but without the source method to enhance the signal at large τ with those from the source method at $\beta = 5.9$ (*i.e.* set 19 versus 11 and 14 in Table 4.2). The result for σa^2 is 0.068(3) versus 0.062(2) and for the 0^{++} mass it is 0.81(3) versus 0.75(6). The estimates by Teper *et.al.* are systematically higher. Two possible reasons are; 1) the results of Teper *et.al.* are not asymptotic *i.e.* they have not been measured at large enough separation or 2) there is a bias in the source method; the effects of the lack of positivity of the transfer matrix have not been fully removed at present separations *i.e.* τ is small in the source method too. More work is needed to decide what is the correct explanation and to get better results.

Estimates of the 2^{++} state are less reliable. The latest word is that $m_{2^{++}}$ is $\approx$ 1.6 $m_{0^{++}}$ [56]. The results of Teper *et.al.* are in agreement with Kolier-Van Baal [62] analytic calculations for small volumes but suggest that continuum behavior only sets in at much larger spatial volumes. The conclusion is that in a small volume the 2^{++} mass is almost degenerate with 0^{++}, but changes rapidly between $10 < L < 16$ at $\beta \approx$ 6.0. The string tension and 0^{++} mass do not show similar strong finite volume effects. Therefore mass ratios are affected and consequently extrapolations cannot be made from small volumes. We need more corraborative data to confirm this exciting result.

The tour de force analytic calculations of Koller and Van Baal [62]have provided an accurate calibration of small volume lattice calculations and an insight to the physical picture in that region. We hope that these calculations can be extended beyond the region where finite size effects are large. At present, we still have to rely on numerical simulations for the continuum behavior. So the focus of current research is to develop update schemes

#	K_F	Lattice	$\sigma(L)L$	$\sqrt{\sigma(\infty)}$	$m_G(L)$	$\frac{m_G(L)}{\sqrt{\sigma(\infty)}}$
1	9.2	$6^3 \times 21$	0.76(5)	0.395(25)	0.91(10)	2.3(3)
2	9.9	$6^3 \times 21$	0.32(1)	0.287(12)	0.79(11)	2.8(4)
3	9.9	$9^3 \times 21$	0.63(2)	0.288(10)	0.89(8)	3.1(3)
4	10.5	$9^3 \times 21$	0.38(1)	0.235(14)	0.67(6)	3.0(3)
5	10.5	$12^3 \times 21$	0.57(3)	0.234(28)	0.64(7)	2.8(4)
6	5.5	$6^3 \times 12$	1.86(6)	0.58(1)	1.07(3)	1.84(6)
7	5.7	$6^3 \times 16$	0.63(2)	0.37(1)	0.66(4)	1.8(1)
8	5.7	$8^3 \times 16$	0.94(3)	0.37(1)	0.86(4)	2.3(1)
9	5.9	$8^3 \times 20$	0.33(1)	0.24(1)	0.73(14)	3.0(6)
10	5.9	$10^3 \times 20$	0.52(1)	0.25(1)	0.68(8)	2.7(4)
11	5.9	$12^3 \times 20$	0.65(3)	0.25(1)	0.74(7)	3.0(3)
12	6.0	$10^3 \times 20$	0.41(3)	0.21(1)	-	-
13	5.9	$10^3 \times 32$	0.48(3)	0.24(3)	0.65(3)	2.7(2)
14	5.9	$12^3 \times 32$	0.66(1)	0.25(1)	0.76(4)	3.05(2)
15	5.9	$16^3 \times 32$	0.86(3)	0.24(2)	0.82(5)	3.4(3)
16	6.0	$13^3 \times 18$	-	-	0.65(8)	-
17	6.05	$13^3 \times 18$	-	-	0.66(7)	-
18	6.1	$13^3 \times 18$	-	-	0.64(10)	-
19	5.9	12^4	-	0.26(1)	0.81(3)	3.1(2)
20	6.0	16^4	-	0.22(1)	0.74(4)	3.3(2)
21	6.2	20^4	-	0.16(2)	0.47(5)	2.9(4)

Table 4.2: Monte Carlo data for the 0^{++} glueball mass and the string tension σ. The first 5 entries are for the improved action calculations [59]. Entries 6 to 12 are from de Forcrand et al [52]; 13 to 15 are from the APE collaboration [55]; 16 to 18 are from DeGrand [61]; and 19 to 21 are from Teper et.al. [56]. Entries 6 to 21 are using the simple Wilson action (i.e. coupling K_F on the Wilson axis).

that have fast decorrelation times [63] , to design better operators as probes of glueballs, to construct sources that enhance the signal at large τ and use improved actions to reduce scaling violations.

4.5) The β-function for Pure Gauge $SU(3)$.

The β-function in QCD measures the rate of change of scale with a change in the coupling constant g. We know the behavior of the β-function at small g from perturbation theory (eqn.(3.4)). At g where lattice calculations are done the 2-lattice MCRG method [2] is a reliable way to measure the discrete non-perturbative β-function of QCD [64] . A test for asymptotic scaling at a given value of β is to compare the MCRG result for $\Delta\beta$ to the 2-loop perturbative result.

On the lattice we measure the discrete β-function; $\Delta\beta$ for a given scale change $b = a_1/a_2$. So far MCRG calculations have used two different scale factors; 1) $b = \sqrt{3}$ and 2) $b = 2$. The present limitation of results with this method is that the starting lattices are not large enough to extract a converged value of $\Delta\beta$; *i.e* show that it is constant over a few blocking levels. Alternately, one can also extract $\Delta\beta$ from physical observables like masses, string tension or the deconfinement transition temperature. The main error in this method is that we have no real estimate of the errors in physical observables *i.e.* have they been obtained from measurements at large seperation? For example, one has yet to show that the string tension is independent of the loop size used in the measurement. The best estimate of the confinement scale is from the measurement of the pure gauge deconfinement transition temperature [65] . Unfortunately, in this case the value of the coupling g is not selected to get a fixed scale change. I have tried to reduce this second error by only using pairs of data points for which the scale change is close to $b = \sqrt{3}$ in fig. 4.1. You should try to estimate what is the possible uncertainty in the global data for $\Delta\beta$ due to these systematic errors. Note that in fig. 4.1, I have rescaled all $\Delta\beta$ to $b = \sqrt{3}$ for comparison.

All calculations agree on the large dip near $\beta = 6$. There is some discrepancy on the approach to the 2-loop perturbative value. The $b = 2$ calculation by the Edinburgh group [66] shows a significant deviation from the perturbative behavior up to $\beta = 7.2$. It would be valuable to reduce the errors at the last two points, $\beta = 7.0$ and 7.2. The $b = \sqrt{3}$ calculation also gives a value smaller than the 2-loop result up to $\beta = 7.0$ [67] . Between $\beta = 6.2$ and 7 there are $\approx 10\%$ deviations from the 2-loop value but $\Delta\beta$ is not as small as the Edinburgh result. The estimates at $\beta = 7.25$ and 7.5 are consistent with asymptotic scaling. The statistics at these two points are 50,000 sweeps (5000 measurements), so the estimate of statistical errors is reliable. The systematic errors are estimated at $\approx 10\%$ over the entire range of β investigated. I feel that the difference in results at large β from using the two renormalization group transformations is significant, so we need to examine the systematics of the two transformations.

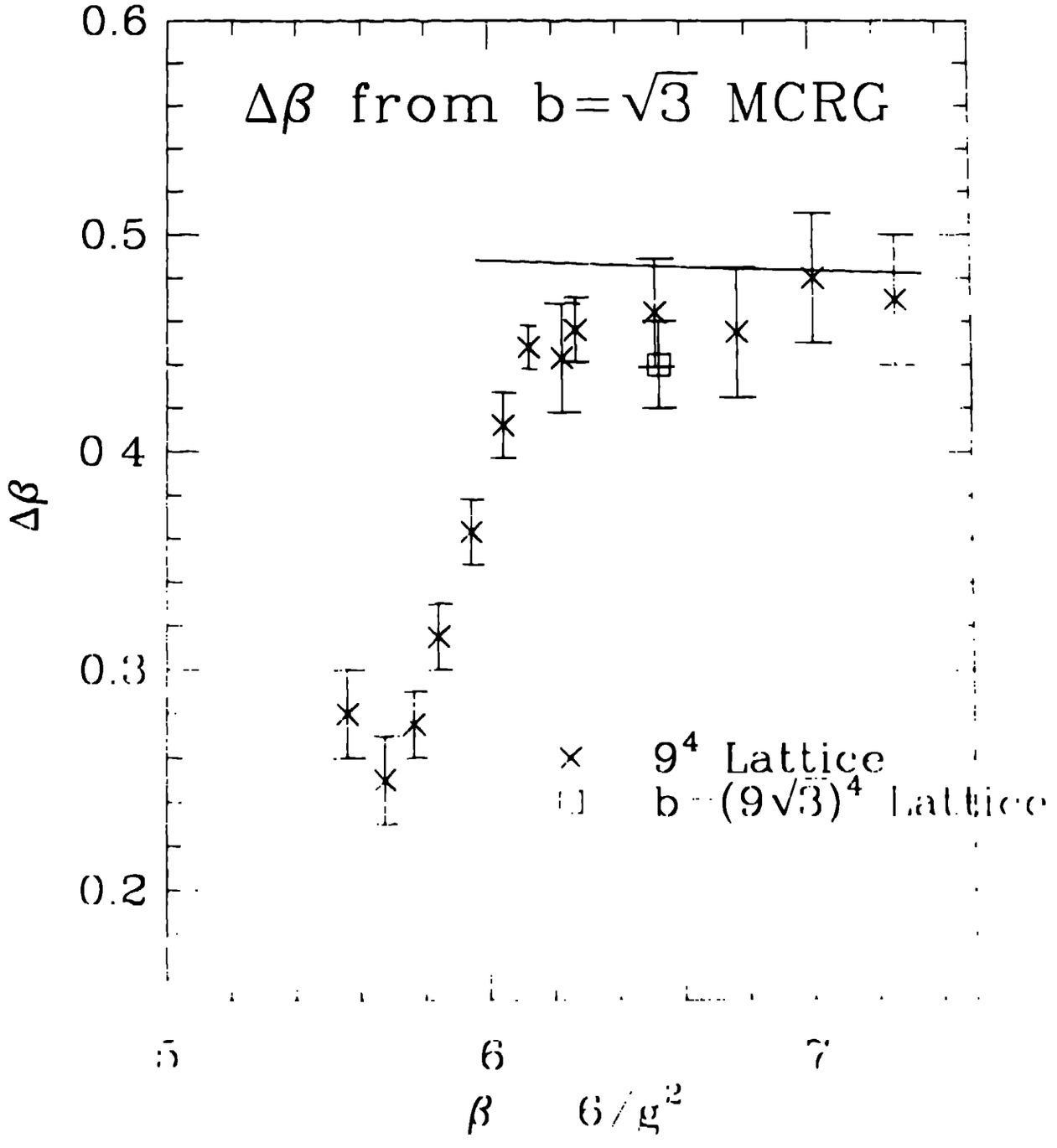
0.6
0.5
0.4
0.3
0.2
$\Delta\beta$ from b=$\sqrt{3}$ MCRG
$\Delta\beta$
9^4 Lattice
b=(9$\sqrt{3}$)^4 Lattice
5
6
7
β 6/g^2

Fig. 4.1

The $\Delta\beta$ result is important because if we want to solve QCD with better than 10% uncertainty, then we need a non-perturbative scaling ansatz since perturbative scaling has at least 10% corrections up to $\beta = 7.0$. The prime source of unknown systematic errors in the MCRG calculation is the number of blocking steps. Because of these, the present calculations cannot distinguish between a slow approach to asymptotic scaling between $\beta = 6.2$ and $\beta = 7.0$ after the pronounced dip at $\beta \approx 6.0$ and the possibility of additional structure due to phase transitions in the extended coupling lattice theory. For the $\sqrt{3}$ transformation, the next obvious refinement step is to redo the calculations with a larger lattice; for example, a $9\sqrt{3}$ lattice allows one more blocking step.

4.6) Status of the Finite Temperature Transition

What happens to Quantum Chromodynamics (QCD) as the temperature is increased? Is there a phase transition to a quark-gluon plasma, and if so what is its order? Over the last couple of years, the answers to these questions for the realistic case of very light u and d quarks and an intermediate mass s quark are beginning to emerge.

Let me first review the expectations for the phase diagram of a theory simpler than QCD: $SU(3)$ gauge theory with n_f degenerate flavors of quarks. The parameters are the quark mass m_q and the temperature T. For $m_q = \infty$ the theory reduces to pure gauge $SU(3)$. There is considerable numerical evidence [68] [65] that this theory has a first order deconfinement transition. The Polyakov line $\langle L \rangle$ is the order parameter for this transition. It vanishes below the transition temperature T_c, but has a non-zero value above. The global $Z(3)$ symmetry of the pure gauge theory is spontaneously broken above T_c. Quarks with finite mass act as external fields coupled to $\langle L \rangle$, and their inclusion weakens the transition. The $Z(3)$ symmetry is explicitly broken, so $\langle L \rangle$ is never zero, but it is still discontinuous at the transition, and thus remains an indicator that the transition is associated with deconfinement. As m_q decreases it is possible that the transition weakens and actually vanishes, allowing an analytic connection between confined and deconfined phases. Also possible is that the transition joins continuously onto the chiral transition, which I discuss next.

For $m_q = 0$ the theory possesses an exact $SU(n_f) \times SU(n_f)$ chiral symmetry. For high T there is an additional approximate axial $U(1)$ symmetry (broken by instanton contributions). It is believed that these symmetries are spontaneously broken at low temperatures, but are restored above a critical temperature. The order parameter for this chiral transition is $\langle \bar{\chi}\chi \rangle$, which is non-zero only below T_c. An ϵ expansion suggests that the transition is first order for $n_f > 3$, but may be first or second order for $n_f = 2$ [69]. For $n_f = 3$ this conclusion is supported by a calculation using an approximate renormalization group equation [70]. At $g = \infty$ (strong coupling expansions), in a Hamiltonian treatment of the $n_f = 2$ theory [71], and in both Hamiltonian and Lagrangian treatments of the $n_f = 4$

theory [72] , the chiral transition is of second order. Turning on m_q removes the transition, and the high and low temperature phases are analytically connected at finite quark mass. As the gauge coupling decreases, however, it is possible that the transition extends out to finite m_q.

For finite m_q, $\langle\bar{\chi}\chi\rangle$ is always non-zero, but may be discontinuous across a transition. If so, this would signify that the transition is associated with chiral symmetry breaking. In fact, if chiral symmetry breaking is driven by confinement, as is suggested by all present evidence, then it may be that, for weak enough gauge coupling, the chiral phase transition line matches onto the deconfinement transition line, cutting the phase plane in two.

The crucial issue for QCD is whether, for quarks with the actual masses observed in nature, the transition is first order, second order or not present at all. The significance of this result is twofold. First, planned heavy ion experiments will be able to investigate hadronic matter at temperatures up to a few hundred MeV. The signature of a transition to a plasma phase depends upon the order of the transition and upon its properties [73] . Second, as the early universe cools down it passes through the transition. If the transition is first order, entropy production reduces the baryon to photon ratio. Furthermore, a first order transition changes the vacuum energy density and thus the cosmological constant. More exotic scenarios, such as neutron-proton separation [74] , also depend crucially on the order and strength of the transition.

Lastly, the study of the chiral transition is an excellent test of the evolving fermion algorithms. There is some controversy in the results from different algorithms which may be due to systematics of the algorithm. Here I will only give the status of the results for $n_f = 2$ and 4.

There is a consensus that for $N_t = 4$ and 4 flavors of staggered fermions, there is a strong first order chiral transition at small masses, *i.e.* $ma \sim 0.025$. On 4^4 lattices this transition has been shown to survives up to $ma = 0.2$ and there is evidence that it exists for all masses [75] . Simulations on a 4×6^3 lattices show that $\langle\bar{\chi}\chi\rangle$ceases to be a good observable to monitor the transition for $ma \gtrsim 0.3$ [76] . However, the discontinuity in $\langle L\rangle$suggests that the transition is still first order for all m.

Data on 4×8^3 lattices with approximate algorithms is debated. Fukugita *et.al.*[77] , show evidence for metastability at $m \sim 0.1$ while Gottlieb *et.al.*[78] and Karsch *et.al.*[79] see a clear signal only at $m = 0.025$. In fig. 4.2, I show results using the Hybrid Monte Carlo algorithm which confirm a first order transition as evidenced by the flip-flops in Wilson loops, $\langle L\rangle$and $\langle\bar{\chi}\chi\rangle$[80] .

Finally, on a 6×10^3 lattice, Kovacs *et.al.*[81] find a first order transition at $ma \sim 0.025$ but not at $ma \sim 0.05$. I believe that a more careful calculation will find metastability at the heavier quark mass.

The status of the chiral transition using approximate algorithms for $n_f \sim 2$ is one

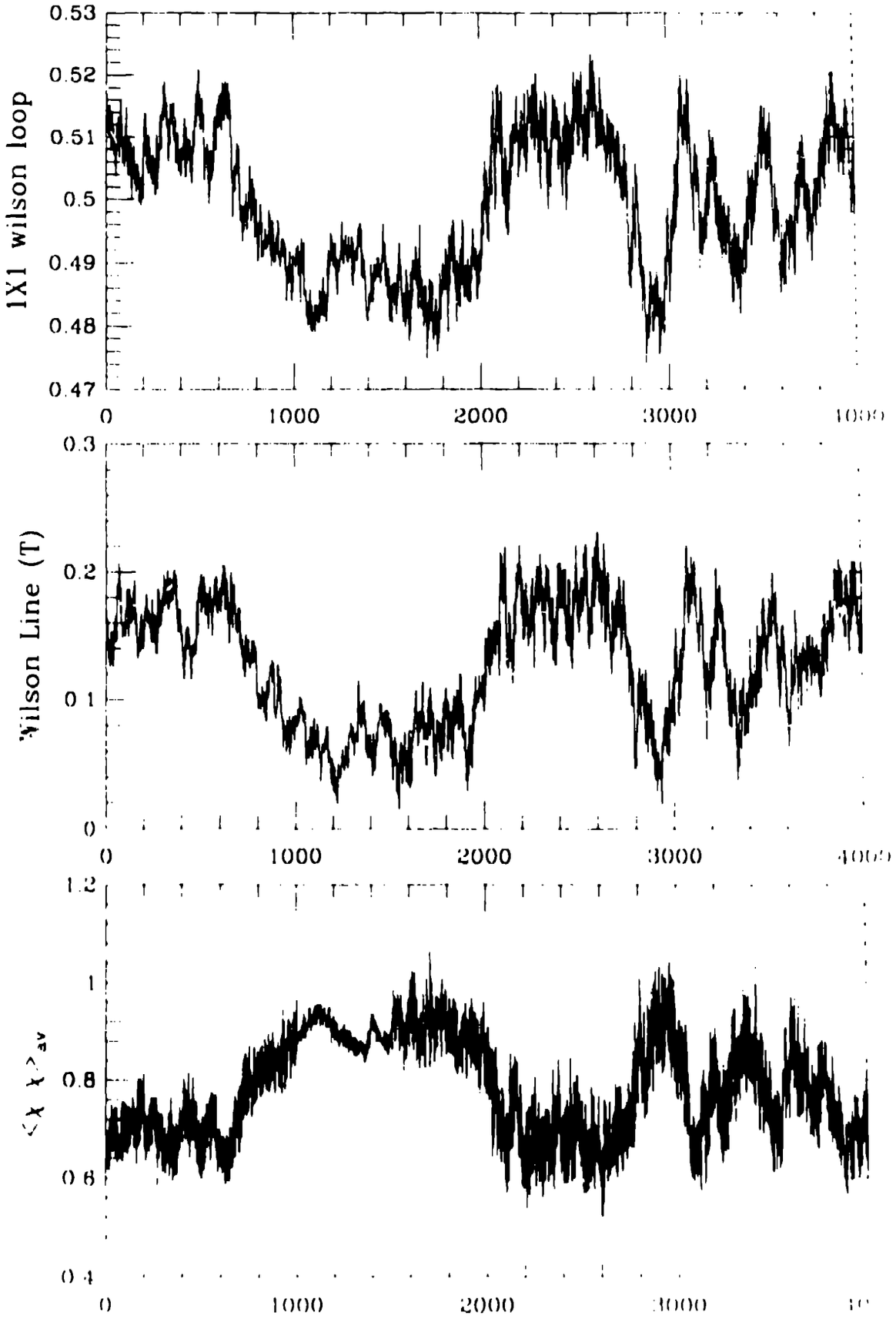
1X1 wilson loop
0.53
0.52
0.51
0.5
0.49
0.48
0.47
0
1000
2000
3000
Wilson Line (T)
0.3
0.2
0.1
0
0
1000
2000
3000
4000
<χ χ>av
1.2
1
0.8
0.6
0.4
0
1000
2000
3000

FIG. 4.2

of conflicting evidence. For $n_f = 2$, Gottlieb *et.al.*[78] see no clear evidence of a transition for $ma = 0.0125 - 0.2$. In contrast, Kogut and Sinclair [82] find a first order transition for $ma = 0.0125$, but no transition for $ma = 0.025$. However, Gavai *et.al.*[83] do find a transition for $m_q a = 0.025$, while Fukugita *et.al.*[84] see one for $ma = 0.1$. We should also mention that Gavai *et.al.*[85] find that the transition is first order along an interpolation between $n_f = 2$ and $n_f = 3$ where two quarks are held at $ma = 0.025$, while the mass of the third is varied between 0.025 and ∞.

Calculations for $n_f = 2$ with an exact algorithm have only been done on a 4^4 lattice at $ma = 0.02$. One again finds a strong first order transition [80].

Recently two independent calculations have further confirmed that at $T > T_c$ chiral symmetry is restored by showing that parity doublet states, (π, σ) and $(N(\frac{1}{2}^+), N^*(\frac{1}{2}^-))$, become degenerate [86] [87]. They also show that the screening lengths for states with these quantum numbers do not become infinite (massless particles) in the limit $m_q \to 0$. We have yet to understand how to connect these screening lengths to physical excitations. So an interesting area for research is to understand the equation of state and properties of matter above T_c and make connections with heavy ion experiments.

The conclusions from present calculations are 1) there is a definite strong first order transition at small quark masses for $n_f = 2, 3, 4$; 2) this transition extends out to sufficiently heavy quark mass to cover the physical case of light u, d and the s quark and 3) the transition temperature is between 100 to 170 *Mev*. To end on an optimistic note, I believe that this estimate will be made more precise in the near future and numerical simulations will provide us with some information on the nature of matter above T_c.

CONCLUSIONS

With the emergence of more powerful computers, development of better algorithms and our ability to formulate the questions better, the estimates of physical quantities from Monte Carlo simulations will progressively get better. I believe that the next five years will be very exciting even though calculations will be long and hard. The treatment in these lectures has been brief and many interesting topics have been skipped. A notable example is strong interaction corrections to weak interaction matrix elements [88]. But, by and large, the greatest deficiency of these lectures is a total lack of discussion of analytic methods. I hope that at least some of you will be motivated to pick up other review articles and probe deeper into the subject.

References

[1] K.G. Wilson, *Phys. Rev.* **D10** (1974) 2445.

[2] K.G. Wilson, *Recent Developments in Gauge Theories*, Cargese (1979), eds. G. t'Hooft *et.al.*(Plenum 1980).

[3] J. Kogut, *Reviews of Modern Physics* **51** (1979) 659.

[4] J. Kogut, *Reviews of Modern Physics* **55** (1982) 775. These two review articles are excellent pedagogical introductions. The details, which I have skipped, will be found here as well as alternate treatments of the same subject matter.

[5] M. Creutz, *Quarks, Gluons and Lattices*, Cambridge Univ. Press 1983.

[6] *Proceedings of the* 1^{st} *CCAST workshop/symposium on "Lattice Gauge Theory using Parallel Processors"*, Beijing 1987, Gordon and Breach 1987.

[7] J.-M. Drouffe and J.-B. Zuber, *Phys. Rep.***102** (1983) 1.

[8] R.P. Feynman and Hibbs, *Path Integrals and Quantum Mechanics*, McGraw Hill 1964.

[9] In principal one can even add irrelevant non gauge invariant operators. However, so far no advantage for doing so has been noted.

[10] M. Lüscher and P. Weisz, *Commun. Math. Phys.* **97** (1985) 59.

[11] A. Patel and R. Gupta, *Phys. Lett.* **183B** (1987) 193.

[12] H. B. Nielsen and M. Ninomiya, *Nucl. Phys.* **B185** (1981) 20.

[13] K. G. Wilson, in *New Phenomena in Subnuclear Physics*, ed. A. Zichichi (plenum Press 1977).

[14] M. Bochicchio, L. Maiani, G. Martinelli, G.C. Rossi and M. Testa, *Nucl. Phys.* **B262** (1985) 331; *Phys. Lett.* **176B** (1986) 445; *Phys. Lett* **178B** (1986) 265; *Phys. Lett.* **181B** (1986) 344; *Nucl. Phys.* **B289** (1987) 505; and *Nucl. Phys.* **B293** (1987) 420.

[15] C. Bernard, T. Draper, A. Soni, D. Politzer and M. Wise, *Phys. Rev.* **D32** (1985) 2343.

[16] J. Kogut and L. Susskind, *Phys. Rev.* **D11** (1975) 395 ;
L. Susskind, *Phys. Rev.* **D16** (1977) 3031.

[17] G. Kilcup and S. Sharpe, *Nucl. Phys.* **B283** (1987) 493.

[18] S. A. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, *Phys. Rev.* **D39** (1987) 2531.

[19] P. Rossi, C.T.H. Davis and G. P. Lepage, UCSD-PTH 87/08.

[20] M. Creutz, *Proceedings of the* 1^{st} *CCAST workshop symposium on Lattice Gauge Theory, Beijing 1987*, Gordon and Breach 1987.

[21] I. Barbour *et.al. Nucl. Phys.* **B275** [FS17] (1986) 296.

[22] B. Svetitsky, *Physics Reports* **132** (1985) 1.

[23] K. G. Wilson and J. B. Kogut, *Phys. Rep.* **12C** (1984) 76.

[24] M. Fisher, Lecture Notes in Physics, Vol 186, Springer-Verlag (1983) ;
P. Pfeuty and G. Toulouse, *Introduction to Renormalization Group and Critical Phenomena*, Wiley 1977.

[25] R. Shrock, in *Proceedings of the 1987 Conference on Field Theory on a Lattice*, Seillac, France.

[26] J. Jersak, in *Proceedings of the 1985 Conference on Lattice Gauge Theories*, Wuppertal, West Germany.

[27] E. Brezin, J.C. le Guillou and J. Zinn Justin, *Phase Transitions and Critical Phenomenon Vol 6*, eds. Domb and Green, Academic Press 1976.

[28] R. Dashen and H. Neuberger, *Phys. Rev. Lett.* **50** (1983) 1897.

[29] A. Hasenfratz *et.al.*, FSU-SCRI-87-52 (1987).

[30] W. Langguth and I. Montvay, DESY-87-20 (1987).

[31] K. Huang, E. Manusakis and J. Polonyi, *Phys. Rev.* **D35** (1987) 3187.

[32] J. Kuti and Yue Shen, UCSD/PTH 87-14.

[33] M. Lüscher and P. Weisz, *Nucl. Phys.* **B290** [FS20] (1987) 25; and DESY-87-075.

[34] R. L. Dobrushin, *Theory Prob. Appl.* **13** (1969) 197. ;
O.E. Lanford III and D. Ruelle, *Commun. Math. Phy.* **13** (1969) 194.

[35] P. Rossi and C.T.H. Davis, UCSD-PTH-11 (July 1987) ;
G. Kilcup and S. Sharpe, Unpublished.

[37] M. Fukugita, *Proceedings of the 1987 meeting on Gauge Theory on a Lattice*, Seillac 1987.

[37] R. Shrock, *Proceedings of the 1987 meeting on Gauge Theory on a Lattice*, Seillac 1987.

[38] R. Gupta, G. Guralnik, G. Kilcup, A. Patel, S. Sharpe and T. Warnock, *Phys. Rev.* **D36** (1987) 2813 (Wilson and Staggered).

[39] K.C. Bowler, R.D. Kenway, D. Roweth and D.B. Stephenson, Edinburgh Preprint 87/424 (Staggered).

[40] P. de Forcrand *et.al.* ~~Wuppertal Preprint, Sept~~ (1987) (Improved Wilson fermion action and blocked gauge configurations).

[41] S. Itoh, Y. Iwasaki, and T. Yoshie, *Phys. Lett.* **183B** (1987) 351; (using Wilson fermions and an improved gauge action).

[42] H. Hamber *Phys. Lett.* **178B** (1986) 277; (Wilson and Staggered fermions).

[43] D. Barkai, K. Moriarty and C. Rebbi, *Phys. Lett.* **156B** (1985) 385; (staggered).

[44] A. Billoire, G. Marinari and R. Petronzio, *Nucl. Phys.* **B251** [FS13] (1985) 141; (Wilson and Staggered).

[45] S. Gottlieb, W. Liu, D. Toussaint, R.L. Renken, and R. Sugar, *Phys. Rev. Lett.* 59 (1987) 1513.

[46] M. Fukugita *et.al. Phys. Rev. Lett.* **57** (1986) 953 (Wilson); *Phys. Lett.* **191B** (1987) 164 (Staggered).
[47] H. Hamber, *Phys. Lett.* **193B** (1987) 292.
[48] M. Campostrini, *et.al. Phys. Lett.* **193B** (1987) 78 (Staggered).
[49] M. Campostrini K. Moriarty and C. Rebbi, Brookhaven Preprint BNL-39777 (1987).
[50] D. Barkai, K. J. M. Moriarty and C. Rebbi, *Phys. Rev.* **D30** (1984) 1292.
[51] S. W. Otto and J. D. Stack, *Phys. Rev. Lett.* **52** (1984) 2328.
[52] P. de Forcrand *et al.*, *Phys. Lett.* **160B** (1985) 137; ;
Ph. de Forcrand, unpublished.
[53] A. Hasenfratz *et.al.*, *Phys. Lett.* **143B** (1984) 193.
[54] P. De Forcrand, *Proceedings of the Metropolis Conference*, Los Alamos (1985), J. Stat. Phys. **43** (1986) 1077.
[55] APE collaboration, Rome Preprint ROM2F/87/014.
[56] M. Teper, *Proceedings of the 1987 meeting on Gauge Theory on a Lattice*, Seillac 1987; ;
C. Michael and M. Teper, Oxford Preprint 72/87.
[57] Ph. de Forcrand, V. Linke and I.O. Stamatescu, Berlin preprint May 1987; ;
M.P. Grady, D.K. Sinclair and J.B. Kogut, Argonne Preprint 87-94.
[58] M. Chanowitz, in *Tsukuba: Hadron 1987*, Page 269.
[59] A. Patel, R. Gupta, G. Guralnik, G. Kilcup and S. Sharpe, *Phys. Rev. Lett.* **57** (1986) 1288.
[60] P. de Forcrand, Unpublished.
[61] T. DeGrand, COLL-HEP-149, and *Phys. Rev.* **D36** (1987) 176.
[62] P. van Baal, and J. Koller, Lattice Conference 1987, Seillac.
[63] The best update method for pure gauge at present is overrelaxation followed by one metropolis hit. For a status see R. Gupta, G. Kilcup, A. Patel and S. Sharpe, LAUR-87-.
[64] For a review see lecture 5 by R. Gupta in Ref. [6].
[65] S. Gottlieb, A.D. Kennedy, J. Kuti, S. Meyer, B.J. Pendleton, R.L. Sugar, and D. Toussaint, *Phys. Rev. Lett.* **55** (1986) 1958, ;
N. Christ and A. Terrano, *Phys. Rev. Lett.* **56** (1986) 111.
[66] K. C. Bowler, A. Hasenfratz, P. Hasenfratz, U. Heller, F. Karsch, R. D. Kenway, I. Montvay, G. S. Pawley, and D. J. Wallace, *Phys. Lett.* **179B** (1987) 375.
[67] R. Gupta, G. Guralnik, A. Patel, T. Warnock and C. Zemach, *Phys. Lett.* **161B** (1985) 352.
[68] J. B. Kogut, H. Matsuoka, S. H. Shenker, J. Shigemitsu, D. K. Sinclair, M. Stone and H. W. Wyld, *Phys. Rev. Lett.* **51** (1983) 869.
[69] R. D. Pisarski and F. Wilczek, *Phys. Rev.* **D2** (1984) 338.

[70] A. Margaritis and A. Patkos, *Phys. Lett.* **178B** (1986) 272.
[71] L. Susskind, *Phys. Rev.* **D16** (1977) 3031.
[72] A. Patel, *Phys. Lett.* **141B** (1984) 244 ;
P. Damgaard, N. Kawamoto and K. Shigemoto, *Phys. Rev. Lett.* **53** (1984) 2211.
[73] B. Svetitsky, *Proceedings of the Fifth International Conference on Ultra-Relativistic Nucleus-Nucleus Collisions, Quark Matter, 86* Asilomar ;
M. Matsua, *ibid.*
[74] J. H. Applegate, C. J. Hogan and R. J. Sc rer, *Phys. Rev.* **D35** (1987) 1151 ;
J. H. Applegate and C. J. Hogan, *Phys. Rev.* **D31** (1985) 3037.
[80] R. Gupta, G. Guralnik, G. Kilcup, A. Patel and S. Sharpe. *Phys. Rev. Lett.* **57** (1986) 2621, and LAUR-87-3068 (Oct 1987).
[76] R. Gupta, in the *Proceedings of the 1987 conference Field Theory on the Lattice*, Seillac, France 1987.
[77] M. Fukugita and A. Ukawa, *Phys. Rev. Lett.* **57** (1986) 503.
[78] S. Gottlieb, W. Liu, D. Toussaint, R.L. Renken, and R.L. Sugar, *Phys. Rev.* **D35** (1987) 3972.
[79] F. Karsch, J. B. Kogut, D. K. Sinclair and H.W. Wyld, *Phys. Lett.* **188B** (1987) 353.
[80] R. Gupta, G. Kilcup, A. Patel and S. Sharpe, *in preparation.*
[81] E.V.E. Kovacs, J. Kogut and D. Sinclair, *Phys. Rev. Lett.* **58** (1987) 751.
[82] J. B. Kogut and D. K. Sinclair, ILL-(TH)-87-46 (September 1987).
[83] R. Gavai, J. Potvin, S. Sanielevici, BNL-40078(1987).
[84] M. Fukugita, S. Ohta, Y. Oyanagi and A. Ukawa, *Phys. Rev. Lett.* **58** (1987) 2515.
[85] R. Gavai, J. Potvin and S. Sanielevici, *Phys. Lett.* **183B** (1987) 86; *Phys. Rev. Lett.* **58** (1987) 2519; and Preprint BNL-87-0742 (September 1987).
[86] C. de Tar and J. Kogut, *Phys. Rev.* **D36** (1987) 2828.
[87] S. Gottlieb *et.al.*, *Phys. Rev. Lett.* **59** (1987) 1881.
[88] For a review see lecture 4 by R. Gupta in Ref. [6].